Constant σ_k-curvature metrics with Delaunay type ends

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Abstract

In this paper we produce families of complete non compact Riemannian metrics with positive constant σ_k-curvature equal to $2^{-k}\binom{n}{k}$ by performing the connected sum of a finite number of given n-dimensional Delaunay type solutions, provided $2 \le 2k < n$. The problem is equivalent to solve a second order fully nonlinear elliptic equation.

Key Words: σ_k-curvature, fully nonlinear elliptic equations, conformal geometry, connected sum

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1 Introduction and statement of the result

In recent years much attention has been given to the study of the Yamabe problem for σ_k–curvature, briefly the σ_k–Yamabe problem. To introduce the analytical formulation, we first recall some background material from Riemmanian geometry. Given (M, g), a compact Riemannian manifold of dimension $n \ge 3$, we denote respectively by Ric_g, R_g the Ricci tensor and the scalar curvature of (M, g). The Schouten tensor of (M, g) is defined as follows

$$A_g \;\; := \;\; \tfrac{1}{n-2} \left(Ric_g \; - \; \tfrac{1}{2(n-1)} R_g g \right) .$$

If we denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the symmetric endomorphism $g^{-1}A_g$, then the σ_k-curvature of (M, g) is defined as the k-th symmetric elementary function of $\lambda_1, \ldots, \lambda_n$, namely

$$\sigma_k(g^{-1}A_g) \;\; := \sum_{i_1 < \ldots < i_k} \lambda_{i_i} \cdot \ldots \cdot \lambda_{i_k} \quad \text{for } 1 \le k \le n \qquad \text{and} \qquad \sigma_0(g^{-1}A_g) \;\; := \;\; 1.$$

The σ_k–Yamabe problem on (M, g) consists in finding metrics with constant σ_k–curvature in the same conformal class of g. The case $k = 1$ is the well known Yamabe problem, whose progressive resolution is due to Yamabe [23], Trudinger [22], Aubin [1] and Schoen [19]. In order to present the existence results for $k \ge 2$, when the equation becomes fully nonlinear, we recall the notion of k-admissibility, which is a

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sufficient condition to insure the ellipticity of the equation. A metric g on M is said to be k–admissible if it belongs to the k–th positive cone Γ_k^+, where

$$g \in \Gamma_k^+ \quad \Longleftrightarrow \quad \sigma_j(g^{-1}A_g) > 0 \quad \text{for} \quad j = 1, \ldots, k.$$

Under the assumption that g is k–admissible the (positive) σ_k–Yamabe problem on closed manifolds has been solved in the case $k = 2$, $n = 4$ by Chang, Gursky and Yang [5] [6], for locally conformally flat manifolds by Li and Li [13] (see also Guan and Wang [8]), and for $2k > n$ by Gursky and Viaclovsky [9]. For $2 \leq 2k \leq n$ the problem has been solved by Sheng, Trudinger and Wang [21] under the extra–hypothesis that the operator is variational. We point out that for $k = 1, 2$ this hypothesis is always fulfilled, whereas for $k \geq 3$ it has been shown in [2] that this extra assumption is equivalent to the locally conformally flatness.

Of interest in this paper is the construction of complete non compact locally conformally flat metrics with constant (positive) σ_k-curvature. These solutions can be regarded as singular solutions for the σ_k-equation on the complement of a discrete set Λ on the standard n-dimensional sphere. To put our result in perspective, we recall that for $k = 1$, the first examples of conformal constant (positive) scalar curvature metrics with isolated singularities have been obtained by Schoen in [20]. Later, Mazzeo and Pacard (see [14] and [15]) have produced different families of solutions on the complement of a singular set Λ consisting of a finite disjoint union of closed smooth submanifolds of arbitrary dimension between 0 and $(n-2)/2$. Another existence result, in the case where Λ is given by an even number of points, is due to Mazzeo, Pollack and Uhlenbeck [16]. We will return on this later, since our construction is closely related to their work. For $2 \leq k < n/2$, the first examples of complete non compact metrics lying in the k-th positive cone and having constant σ_k-curvature have been obtained by the first author in a joint work with Ndiaye [17], assuming that the points of the singular set have a symmetric disposition.

Some comments about the asymptotic behavior of the singular solutions are now in order. For $k = 1$ it follows from the works of Caffarelli, Gidas and Spruck [3] and Korevaar, Mazzeo, Pacard and Schoen [12] that every complete non compact locally conformally flat metric with constant positive scalar curvature must be asymptotic to a radial solution. In a recent work, Han, Li and Teixeira [10] have shown that this fact is true also for metrics of constant σ_k curvature lying in the k-th positive cone, provided $2 \leq k < n/2$. Notice that for $k \geq n/2$ the singularity is always removable. For these reason, it is clear that complete radial solutions are going to play a fundamental role in our construction. These particular solutions are also known as Delaunay-type metrics and have been classified by Chang, Han and Yang in [7] and we recall them briefly in Section 3. Essentially, they are conformally cylindrical metrics with a periodic conformal factor, whose minimum will be referred as Delaunay-parameter.

As anticipated, the construction presented in this paper is inspired by [16] and consists in performing the connected sum of a finite number of Delaunay-type metrics. The solutions obtained in this way are quite different from the ones produced in [17], which roughly speaking looks like a spherical central body with several Delaunay-type ends having small Delaunay parameters. In the present construction, the Delaunay parameters are not forced to be small, hence our solutions may possibly belong to a different connected component of the moduli space.

To fix the notations, we recall that the connected sum of two n–dimensional Riemannian manifolds (D_1, g_1) and (D_2, g_2) is the topological operation which consists in removing an open ball from both D_1 and D_2 and identifying the leftover boundaries, obtaining a new manifold with possibly different topology. Formally, if $p_i \in D_i$ and for a small enough $\varepsilon > 0$ we excise the ball $B(p_i, \varepsilon)$ from D_i, $i = 1, 2$, the (pointwise) connected sum M_ε of D_1 and D_2 along p_1 and p_2 with *necksize* ε is the topological manifold defined as

$$M_\varepsilon \;:=\; D_1 \sharp_\varepsilon D_2 \;=\; \left[D_1 \setminus B(p_1, \varepsilon) \,\cup\, D_2 \setminus B(p_2, \varepsilon) \right] \big/ \sim \,,$$

where $\sim$ denotes the identification of the two boundaries $\partial B(p_i, \varepsilon)$, $i = 1, 2$. Of course the new manifold M_ε can be endowed with both a differentiable structure and a metric structure, as it will be explicitly

done in Section 4. Even though from a topological point of view the value of the *necksize* is forgettable, it will be important to keep track of it, when we will deal with the metric structure.

Concerning the solvability of the Yamabe equation ($k = 1$) on the pointwise connected sum of manifolds with constant scalar curvature, we recall the results of Joyce [11] for the compact case and the already mentioned work of Mazzeo, Pollack and Uhlenbeck [16] for the non compact case. For $2 \leq k < n/2$ and compact manifolds, a connected sum result has been provided by the first author in a joint work with Catino [4]. Our main result here is the following

Theorem 1. *Let $(D_1, g_1), \dots, (D_N, g_N)$ be a collection of n-dimensional Delaunay-type solutions (see Proposition 3.1) to the positive σ_k-Yamabe problem, with $2 \leq 2k < n$. Then, there exists a positive real number $\varepsilon_0 > 0$ only depending on n, k, and the C^2–norm of the coefficients of g_1 and g_2 such that, for every $\varepsilon \in (0, \varepsilon_0]$, the connected sum $M_\varepsilon = D_1 \sharp_\varepsilon \dots \sharp_\varepsilon D_N$ can be endowed with a metric $\widetilde{g}_\varepsilon$ with constant σ_k–curvature equal to $2^{-k}\binom{n}{k}$. Moreover $\|\widetilde{g}_\varepsilon - g_i\|_{C^r(K_i)} \to 0$ for any $r > 0$ and any compact set $K_i \subset D_i \setminus \{p_i\}$, the p_i's, $i = 1, \dots, N$, being the points about which the connect sum is performed.*

Some comments about the strategy of the proof are in order. Incidentally, we notice that the constant $2^{-k}\binom{n}{k}$ arises naturally as the σ_k–curvature of the n-dimensional standard sphere, so we will end up with a family of metrics $\{\widetilde{g}_\varepsilon\}_\varepsilon$ parametrized in terms of the *necksize* which satisfy

$$\sigma_k\big(\widetilde{g}_\varepsilon^{-1} A_{\widetilde{g}_\varepsilon}\big) \quad = \quad 2^{-k}\binom{n}{k}. \tag{1.1}$$

To show the existence of these solutions, we start by writing down (see Section 4) an explicit family of approximate solution metrics $\{g_\varepsilon\}_\varepsilon$ (still parametrized by the *necksize*) on M_ε. This metrics are complete and non compact, since they coincide with the original Delaunay-type metrics g_i on $D_i \setminus B(p_i, \varepsilon)$, $i = 1, 2$, and are close to a model metric on the remaining piece of the connected sum manifold, which in the following will be referred as neck region. The metric which we are going to use as a model in the neck region is described in Section 3. It is a complete metric on $\mathbb{R} \times \mathbb{S}^{n-1}$ with zero σ_k–curvature and yields a natural generalization of the scalar flat Schwarzschild metric. It has been successfully employed in [4] to treat the connected sum of constant scalar curvature manifolds and for the local analysis on the neck region we will refer to this work.

The next step in our strategy amounts to look for a suitable correction of the approximate solutions to the desired exact solutions. This will be done by means of a global conformal perturbation. At the end it will turn out that for sufficiently small values of the parameter ε such a correction can actually be found together with a very precise control on its size and this will ensure the smooth convergence of the new solutions $\tilde{g}_\varepsilon$ to the former metrics g_i on the compact subsets of $M_i \setminus \{p_i\}$, $i = 1, 2$. We point out that it is also important to control the asymptotic behavior of such a perturbation in order to preserve the completeness of the approximate solutions. Typically, one is led to search for corrections which present a decay at infinity.

The main point in the correction procedure is to provide invertibility for the linearized operator about the approxiamte solutions, together with uniform (with respect to the *necksize* parameter ε) *a priori* bounds. This will enable us to carry out the perturbative nonlinear analysis (Section 7) by proving the convergence of a Newton iteration scheme. The uniformity of the *a priori* bound will follow from the use of weighted function spaces with a weighting function acting on the neck region, in analogy with the analysis contained in [4]. On the other hand, the invertibility issue is quite different from the compact case. In fact, in order to obtain the desired Fredholm properties for the linearized operator, we will further introduce weighting functions with gradient supported outside of a compact region of M_ε. In analogy with the case $k = 1$ (see [16]), the analysis is complicated by the lack of coercivity of the linearized operator. This is due to the conformal invariance of the σ_k-equation. In fact, the functions which are responsible for this lack of coercivity arise as infinitesimal generators (Jacobi fields) of conformal transformations. As it will be made clear in Section 7, the geometrical interpretation of the Jacobi fields will be exploited in order to insure the completeness, after the perturbation, of the exact solutions.

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2 Notations and preliminaries

We fix now the notations that will be used throughout this paper. Let $(M, \bar{g})$ be a compact smooth n–dimensional Riemannian manifold without boundary an let $2 \leq 2k < n$. Taking advantage of this second assumption, we introduce the following formalism for the conformal change

$$\bar{g}_u \quad := \quad u^{\frac{4k}{n-2k}}\, \bar{g},$$

where the conformal factor $u > 0$ is a positive smooth function. In this context $\bar{g}$ will be referred as the background metric. At a first time the σ_k–equation for the conformal factor u can be formulated as

$$\sigma_k\big(\, \bar{g}_u^{-1} A_{\bar{g}_u}\big) \quad = \quad 2^{-k} \tbinom{n}{k}.$$

We recall that the Schouten tensor of $\bar{g}_u$ is related to the one of $A_{\bar{g}}$ by the conformal transformation law

$$A_{\bar{g}_u} \quad = \quad A_{\bar{g}} - \tfrac{2k}{n-2k} u^{-1} \nabla^2 u + \tfrac{2kn}{(n-2k)^2} u^{-2} du \otimes du - \tfrac{2k^2}{(n-2k)^2} u^{-2} |du|^2 \bar{g},$$

where ∇^2 and $|\cdot|$ are computed with respect to the background metric $\bar{g}$. For technical reasons, it is convenient to set

$$B_{\bar{g}_u} \quad := \quad \tfrac{n-2k}{2k}\, u^{\frac{2n}{n-2k}}\, \bar{g}_u^{-1} \cdot A_{\bar{g}_u}$$

and to reformulate the σ_k–equation as

$$\mathcal{N}(u, \bar{g}) \quad := \quad \sigma_k\left(B_{\bar{g}_u}\right) - \tbinom{n}{k}\big(\tfrac{n-2k}{4k}\big)^k u^{\frac{2kn}{n-2k}} \quad = \quad 0. \tag{2.1}$$

We notice that if two metrics $\bar{g}$ and g are related by $\bar{g} = (v/u)^{4k/(n-2k)} g$, then the nonlinear operator enjoys the following *conformal equivariance property*

$$\mathcal{N}\left(u, \bar{g}\right) \quad = \quad (v/u)^{-\frac{2kn}{n-2k}}\, \mathcal{N}\left(v, g\right). \tag{2.2}$$

The linearized operator of $\mathcal{N}(\,\cdot\,, \bar{g})$ about u is defined as

$$\mathbb{L}(u, \bar{g})\,[w] \quad := \quad \frac{d}{ds}\bigg|_{s=0} \mathcal{N}\left(u + sw, \bar{g}\right). \tag{2.3}$$

Most part of the analysis in this paper (Sections 5 and 6) is concerned with the study of the mapping properties of the linearized operator about the approximate solutions g_ε's, that we will write in the form $u_\varepsilon^{4k/(n-2k)}\bar{g}$. As a direct consequence of the property (2.2), we have the following *conformal equivariance property* for the linearized operator

$$\mathbb{L}(u, \bar{g})[w] \;=\; (v/u)^{-\frac{2kn}{n-2k}}\, \mathbb{L}(v, g)[(v/u)\, w]. \tag{2.4}$$

3 Delaunay and Schwarzschild type metrics on $\mathbb{R} \times \mathbb{S}^{n-1}$

We start this section with the description of a particular family of complete metrics on the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ with constant σ_k-curvature equal to $2^{-k}\binom{n}{k}$. These metrics are conformal to the cylindrical one g_{cyl} on the whole cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ (notice that in the following, the cylindrical metric will also be denoted by $g_{cyl} = dt^2 + d\theta^2$, where $d\theta^2$ represents the standard metric on $\mathbb{S}^{n-1}$).

Let us consider then on the standard cylinder $(\mathbb{R} \times \mathbb{S}^{n-1}, dt^2 + d\theta^2)$ a conformal metric g of the form $g = v^{4k/(n-2k)}\, g_{cyl}$, where the conformal factor v only depends on the t variable, i.e., $v = v(t)$, and let us impose the condition $\sigma_k\,\left(g^{-1}A_g\right) = 2^{-k}\binom{n}{k}$ or equivalently

$$\mathcal{N}(v, g_{cyl}) = 0\,. \tag{3.1}$$

It is easy to observe that, under the usual change of coordinates, $t = -\log|x|$ and $\theta = x/|x|$, this corresponds to look for a metric on $\mathbb{R}^n \setminus \{0\}$ which has constant positive σ_k-curvature and which is radially symmetric. These metrics has been studied in [7] by Chang, Han and Yang and we refer the reader to their work for further details. Here we just recall the following

Proposition 3.1 (Delaunay-type metrics)**.** *Let g_v be a a metric on $\mathbb{R}^n$ of the form $g_v = v^{4k/(n-2k)}\, g_{cyl}$, where v is a smooth positive function only depending on the variable $t \in \mathbb{R}$. Let us define the quantity*

$$H\,(v, \dot v) \;:=\; \left[\, v^2 \,-\, \left(\tfrac{2k}{n-2k}\right)^2 \dot v^2 \,\right]^k \,-\, v^{\frac{2kn}{n-2k}}$$

Then, if $H\,(v,\dot v) \,\equiv\, H_0 \in \left(0\,,\, \frac{2k}{n-2k}\left(\frac{n-2k}{n}\right)^{n/2k}\right)$, in correspondence of each H_0, there exists a unique solution v to

$$\left[\, v^2 \,-\, \left(\tfrac{2k}{n-2k}\right)^2 \dot v^2 \,\right]^{k-1} \left[\, v \,-\, \left(\tfrac{2k}{n-2k}\right)^2 \ddot v \,\right] \;=\; \tfrac{n}{n-2k}\, v^{\frac{2kn}{n-2k}-1}\,. \tag{3.2}$$

satisfying the conditions $\dot v(0) = 0$, and $\ddot v(0) > 0$. This family of solutions gives rise to a family of complete and periodic metrics on $\mathbb{R} \times \mathbb{S}^{n-1}$ satisfying

$$\sigma_k\,\left(\,B_{g_v}\,\right) \,=\, 2^{-k}\binom{n}{k} \quad in\ \mathbb{R} \times \mathbb{S}^{n-1} \tag{3.3}$$

This solution is periodic and it is such that $0 < v(t) < 1$ for all $t \in \mathbb{R}$. In the following we will index the conformal factors and the metrics in this family by means of the parameter $\eta := v(0)^{2k/(n-2k)}$ which represents the neck-size. Notice that $0 < \eta < \left(\frac{n-2k}{n}\right)^{1/2k}$ and that the period of $v_{D,\eta}$ will be denoted by T_η.

As anticipated in the introduction, the first step in our strategy amounts to build approximate solutions on the connected sum of a finite number of Delaunay type solutions with possibly different neck-size parameter $(D_{\eta_1}, g_{\eta_1}), \ldots, (D_{\eta_N}, g_{\eta_N})$. To this end, we need to modify the original metrics in a neighborhood of the points that we are going to excise, obtaining a new metric in the so called *neck region*. In the scalar curvature case ($k = 1$), a clever choice turns out to be the (space-like) *Schwarzschild* metric. This is a complete scalar flat metric conformal to the cylindrical metric g_{cyl} on $\mathbb{R} \times \mathbb{S}^{n-1}$. The explicit formula is given by

$$g \;:=\; \cosh\left(\tfrac{n-2}{2}t\right)^{\frac{4}{n-2}}\, g_{cyl}\,.$$

In a similar way, it is easy to construct a complete conformal metric on $\mathbb{R} \times \mathbb{S}^{n-1}$ with zero σ_k–curvature, for all $2 \le 2k < n$. We have

Proposition 3.2 (Schwarzchild metrics)**.** *Let g_v be a metric on $\mathbb{R} \times \mathbb{S}^{n-1}$ of the form $g_v = v^{4k/(n-2k)} g_{cyl}$, where v is a positive smooth function depending only on $t \in \mathbb{R}$. Let us define the quantity*

$$h(t) := v^2(t) - \left(\tfrac{2k}{n-2k}\right)^2 \dot v^2(t).$$

Then, if $h_0 := h(0) > 0$, the family of positive solutions to the equation

$$\sigma_k(B_{g_v}) = 0 \quad in\ \mathbb{R} \times \mathbb{S}^{n-1}$$

is given by $v(t) = \sqrt{h_0}\cosh\left(\frac{n-2k}{2k}t - c\right)$, $c \in \mathbb{R}$. In the following, the solution with $c = 0$ will be denoted by v_Σ.

For sake of completeness and for future convenience, we recover from [4] the following formula for the linearized σ_k-operator about the Schwarzschild type metric.

Lemma 3.3. *The linearized σ_k-operator about the σ_k-Schwarzschild metric*

$$\mathbb{L}^0(v_\Sigma, g_{cyl})[w] := \frac{d}{ds}\bigg|_{s=0} \sigma_k\left(B_{g_s}\right),$$

where $g_s = g_{v+sw}$, is given by

$$\mathbb{L}^0(v_\Sigma, g_{cyl})[w] = -C_{n,k}\, v_\Sigma\, h_\Sigma^{k-1}\left[\,\partial_t^2 + \tfrac{n-k}{k(n-1)}\Delta_\theta - \left(\tfrac{n-2k}{2k}\right)^2\right] w\,, \tag{3.4}$$

where $C_{n,k} \;=\; \binom{n-1}{k-1}\;\left(\frac{n-2k}{4k}\right)^{k-1}$.

Incidentally, we note that the computation (see [4]) leading to (3.4) also shows that

$$\sigma_{k-1-j}(B_0) \;=\; \left(\tfrac{n-2k}{4k}\right)^{k-1-j}\; h_\Sigma^{k-1-j}\;\tfrac{1+j}{k}\,\binom{n}{k-1-j}. \tag{3.5}$$

From this it follows that the σ_k–Schwarzschild metric g_Σ belongs to $\overline{\Gamma}_k^+ \cap \Gamma_{k-1}^+$, for $2 \leq 2k < n$.

4 Approximate solutions

In this section we first describe the construction of the connected sum of a finite number of Delaunay-type solutions $D_{\eta_1}, \ldots, D_{\eta_N}$ and then we define on this new manifold a family of metrics which will represent the approximate solutions to our problem.

Since the whole construction is local, we restrict ourself to the connected sum of two Delaunay type solutions (D_{η_1}, g_1) and (D_{η_2}, g_2). In the following we will denote by $M_\varepsilon := D_{\eta_1}\sharp_\varepsilon D_{\eta_2}$ the manifold obtained by excising two geodesic balls of radius $\varepsilon \in (0,1)$ centered at $p_1 \in D_{\eta_1}$ and $p_2 \in D_{\eta_2}$ and identifying the two left over boundaries. The manifold D_{η_1} and D_{η_2} are endowed with the metrics

$$g_1 \;=\; v_{D,\eta_1}^{\frac{4k}{n-2k}}(dr_1^2 + g_{\mathbb{S}^{n-1}}) \qquad \text{and} \qquad g_2 \;=\; v_{D,\eta_2}^{\frac{4k}{n-2k}}(dr_2^2 + g_{\mathbb{S}^{n-1}})\,,$$

respectively. Starting from g_1 and g_2, we will define on M_ε a new metric g_ε which agrees with the old ones outside the balls of radius one around p_1 and p_2 and which is modeled on (a scaled version of) the σ_k–Schwarzschild metric in the neck region.

To describe the construction, we consider the diffeomorphisms given by the exponential maps

$$\exp_{p_i} : B(O_{p_i}, 1) \subset T_{p_i}D_{\eta_i} \longrightarrow B(p_i, 1) \subset D_{\eta_i}, \quad i = 1, 2.$$

Next, to fix the notations, we identify the tangent spaces $T_{p_i}D_{\eta_i}$ with $\mathbb{R}^n$. It is well known that this identification yields normal coordinates centered at the points p_i, namely

$$x : B(p_1, 1) \longrightarrow \mathbb{R}^n \qquad \text{and} \qquad y : B(p_2, 1) \longrightarrow \mathbb{R}^n.$$

We introduce now asymptotic cylindrical coordinates on the punctured ball $B^*(0,1) = x\left(B(p_1,1)\setminus\{p_1\}\right)$ setting $t := \log\varepsilon - \log|x|$ and $\theta := x/|x|$. In this way we have the diffeomorphism $B^*(0,1) \simeq (\log\varepsilon, +\infty)\times \mathbb{S}^{n-1}$. Analogously, we consider the diffeomorphism $y\left(B(p_2,1)\setminus\{p_2\}\right) = B^*(0,1) \simeq (-\infty, -\log\varepsilon)\times\mathbb{S}^{n-1}$, this time setting $t := -\log\varepsilon + \log|y|$ and $\theta := y/|y|$.

In order to define the differential structure of M_ε, we excise a geodesic ball $B(p_i,\varepsilon)$ from D_{η_i}, obtaining an annular region $A(p_i,1,\varepsilon) := B(p_i,1)\setminus B(p_i,\varepsilon)$, $i = 1,2$. The asymptotic cylindrical coordinates introduced above can be used to define a natural coordinate system on the neck region

$$(t,\theta) :\; \left[A(p_1,1,\varepsilon)\sqcup A(p_2,1,\varepsilon)\right]/\sim\; \longrightarrow (\log\varepsilon, -\log\varepsilon)\times\mathbb{S}^{n-1} =: N_\varepsilon,$$

where $\sim$ denotes the relation of equivalence which identifies the boundaries of $B(p_1,\varepsilon)$ and $B(p_2,\varepsilon)$, namely

$$q_1 \sim q_2 \iff x/|x|(q_1) = y/|y|(q_2) \quad \text{and} \quad |x|(q_1) = \varepsilon = |y|(q_2).$$

Clearly, in this coordinates, the two identified boundaries correspond now to the set $\{0\} \times \mathbb{S}^{n-1}$. To complete the definition of the differential structure of M_ε it is sufficient to consider the old coordinate charts on $D_{\eta_i} \setminus B(p_i, 1)$, $i = 1, 2$.

We are now ready to define on M_ε the approximate solution metric g_ε. First of all, we define g_ε to be equal to the g_i on $D_{\eta_i} \setminus B(p_i, 1)$, $i = 1, 2$. To define g_ε in the neck region, we start by observing that the choice of the normal coordinate system allows us to expand the two metric g_1 and g_2 around p_1 and p_2 respectively as

$$g_1 = \big[\delta_{\alpha\beta} + \mathcal{O}\big(|x|^2\big)\big]\; dx^\alpha \otimes dx^\beta \quad \text{and} \quad g_2 = \big[\delta_{\alpha\beta} + \mathcal{O}\big(|y|^2\big)\big]\; dy^\alpha \otimes dy^\beta .$$

Recalling that the metrics g_1 and g_2 are *locally conformally flat* and using the (t,θ)-coordinates introduced above, we can write

$$\begin{aligned} g_1 &= u_1^{\frac{4k}{n-2k}} (1+c_1)(dt^2 + d\theta^2), \quad \text{with } u_1(t) := \varepsilon^{\frac{n-2k}{2k}} e^{-\frac{n-2k}{2k} t} \\ g_2 &= u_2^{\frac{4k}{n-2k}} (1+c_2)(dt^2 + d\theta^2), \quad \text{with } u_2(t) := \varepsilon^{\frac{n-2k}{2k}} e^{\frac{n-2k}{2k} t} \end{aligned}$$

Now, we fix as background metric on M_ε the following

$$\bar{g} := \begin{cases} g_i & \text{on } D_{\eta_i} \setminus B(p_i, 1) \\ (1+c)(dt^2 + d\theta^2) & \text{on } A(p_1, 1, \varepsilon) \sqcup A(p_2, 1, \varepsilon)]/\sim \end{cases}$$

where

$$c \;:=\; \eta\, c_1 + (1-\eta)\, c_2,$$

with η a smooth and non decreasing cut off function such that $\eta : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ and identically equal to 1 in $(\log\varepsilon, -1]$ and 0 in $[1, -\log\varepsilon)$. Subsequentely, we consider another non increasing smooth function $\chi : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ which is identically equal to 1 in $(\log\varepsilon, -\log\varepsilon - 1]$ and which satisfies $\lim_{t\to -\log\varepsilon} \chi = 0$. Using these cut-off functions, we can now define a new conformal factor

$$u_\varepsilon \;:=\; \begin{cases} 1 & \text{on } D_{\eta_i} \setminus B(p_i, 1) \\ \chi(t)\, u_1 + \chi(-t)\, u_2 & \text{on } A(p_1, 1, \varepsilon) \sqcup A(p_2, 1, \varepsilon)]/\sim \end{cases} \tag{4.1}$$

Finally we define on M_ε the family of approximate solution metrics g_ε, by setting

$$g_\varepsilon \;:=\; u_\varepsilon^{\frac{4k}{n-2k}}\, \bar{g}\ . \tag{4.2}$$

To conclude this section, we observe that with this definition we immediately have that for every $m \in \mathbb{N}$ the approximate solution metrics converge to g_i on the compact subset of $D_{\eta_i} \setminus \{p_i\}$ with respect to the C^m-topology when the parameter ε tends to 0, for $i = 1, 2$. For these reasons, we expect that the size of the term $\mathcal{N}(u_\varepsilon, \bar{g})$, which represents the fail of u_ε from being an exact solution, will become smaller and smaller when $\varepsilon \to 0$. Finally, we notice that adapting the proof of [4, Lemma 3.2] it is straightforward to show that for ε sufficiently small g_ε lies in Γ_{k-1}^+.

5 Analysis of the linearized operator about the Delaunay-type metrics

In this section we discuss some boundary value problems for the linearized operator introduced in (2.3) about Delaunay-type metrics. This local analysis, will find its application in Section 6.

We start by recovering from [17] the expression for the linearized operator about a Delaunay-type metric $g_{D,\eta} = v_{D,\eta}^{4k/(n-2k)}\, g_{cyl}$. We set

$$h \;:=\; v_{D,\eta}^2 \,-\, \left(\tfrac{2k}{n-2k}\right)^2 \dot v_{D,\eta}^2 \qquad \text{and} \qquad F \;:=\; v_{D,\eta}^{\frac{2kn}{n-2k}} \big/ \big(H \,+\, v_{D,\eta}^{\frac{2kn}{n-2k}} \big) \tag{5.1}$$

and we recall that that $h(t) \,>\, 0$ for any $t \in \mathbb{R}$ (see [7] and [17]). With these definitions at hand, we can state the following

Lemma 5.1 (Linearization about the Delaunay-type metrics)**.** *The linearized operator about the Dealunay-type solution $v_{D,\eta}$ is given by*

$$\mathbb{L}\,(v_{D,\eta}\,, g_{cyl})\,[w] \quad = \quad -\; C_{n,k}\; v_{D,\eta}\; h^{\frac{k-1}{2}} \;\; \big\{\, \partial_t^2 \,+\, a_\eta\, \Delta_\theta \,-\, p_\eta \,\big\} \;\; [h^{\frac{k-1}{2}}\, w]\;, \tag{5.2}$$

where Δ_θ is the Laplace-Beltrami operator for standard round metric on $g_{\mathbb{S}^{n-1}}$ and the coefficients a_η and p_η are given by

$$a_\eta \quad := \quad \tfrac{n-k}{k(n-1)} \;+\; \tfrac{n(k-1)}{k(n-1)}\; F\;, \tag{5.3}$$

$$\begin{aligned} p_\eta \quad := \quad & \left(\tfrac{n-2k}{2k}\right)^2 \,+\, \frac{n\,(nk+n-2k)\,(k-1)}{2k^2}\, F \,-\, \frac{n^2\,(k^2-1)}{4k^2}\, F^2 \\ & -\, \frac{n\,(2kn-n+2k)}{4k}\, v_{D,\eta}^{\frac{4k}{n-2k}}\, F^{\frac{k-1}{k}} \,+\, \frac{n^2\,k\,(k-1)}{4k^2}\, v_{D,\eta}^{\frac{4k}{n-2k}}\, F^{\frac{2k-1}{k}}\;. \end{aligned} \tag{5.4}$$

and the constant $C_{n,k}$ is defined by $C_{n,k} \,:=\, \binom{n-1}{k-1}\, \left(\frac{n-2k}{4k}\right)^{k-1}$. For notational convenience we also define the conjugate linearized operator by

$$\mathcal{L}_\eta \quad := \quad \partial_t^2 \,+\, a_\eta\, \Delta_\theta \,-\, p_\eta\;. \tag{5.5}$$

Moreover, we have that there exists a positive constant $c = c(n,k) > 0$ such that for every $j \geq n+1$ and every admissible Delaunay parameter η

$$a_\eta\, \lambda_j \,+\, p_\eta \quad \geq \quad c\;, \tag{5.6}$$

where the positive real numbers λ_j, $j \in \mathbb{N}$, denote the eigenvalues (counted with multiplicity) of Δ_θ, i.e., $-\Delta_\theta \phi_j \,=\, \lambda_j\, \phi_j$.

As a consequence of the last inequality, we will obtain the coercivity of the conjugate linearized operator $\mathcal{L}_\eta$ along the high frequencies (i.e., for $j \geq n+1$).

5.1 Jacobi fields

Using the conformal equivariance of the equation we introduce new families of solutions which are variations of the standard Delaunay solution with neck-size parameter η. The infinitesimal generators of these variations will provide us with natural elements sitting in the kernel of the linearized operator $\mathbb{L}(v_{D,\eta}, g_{cyl})$ about $v_{D,\eta}$, namely the Jacobi fields.

The first remark is that since the equation (3.2) is autonomous, then the solutions are translation invariant (with respect to the t variable). In particular, for $\tau > 0$, the functions $v_{D,\eta,\tau}(t)$, defined by

$$v_{D,\eta,\tau}(t) \quad := \quad v_{D,\eta}(\,t + \log(\tau+1))\;, \tag{5.7}$$

are still solution to (3.2). To find other possible families of solutions it is convenient to use the conformal equivariance of equation (3.3). First notice that, writing $t = -\log|x|$ and $\theta = x/|x|$, with $x \in \mathbb{R}^n \setminus \{0\}$, the cylindrical metric and the Euclidean one are related by $g_{cyl} = |x|^{-2} g_{\mathbb{R}^n}$ on $\mathbb{R}^n \setminus \{0\}$. As a consequence of (2.2) we get

$$\mathcal{N}\,(v, g_{cyl}\,) \quad = \quad |x|^n\, \mathcal{N}\,(\,|x|^{-\frac{n-2k}{2k}}\, v, g_{\mathbb{R}^n})\;. \tag{5.8}$$

Hence, if $v(t,\theta)$ solves $\mathcal{N}(v, g_{cyl}) = 0$ on $\mathbb{R}\times\mathbb{S}^{n-1}$, then the function $u(x)$, defined on $\mathbb{R}^n\setminus\{0\}$ by

$$u(x) \quad := \quad |x|^{-\frac{n-2k}{2k}}\, v(-\log|x|\,,\, x/|x|)\;, \tag{5.9}$$

is a solution to $\mathcal{N}(u, g_{\mathbb{R}^n}) = 0$ on $\mathbb{R}^n\setminus\{0\}$. In particular, the Delaunay solutions $v_{D,\eta}(t)$ defined on the cylinder correspond to the radial solutions of the latter equation $u_{D,\eta}(|x|) := |x|^{-(n-2k)/2k}\, v_{D,\eta}(-\log|x|)$ with a pole in the origin. Since the equation satisfied by $u_{D,\eta}$ is clearly translation invariant (due to the fact that the background metric is $g_{\mathbb{R}^n}$), we have that, for $b\in\mathbb{R}^n$, the n-parameter family of functions $u_{D,\eta,b}(x)$, defined by

$$u_{D,\eta,b}(x) \quad := \quad u_{D,\eta}(|x-b|) \quad = \quad |x-b|^{-\frac{n-2k}{2k}} v_{D,\eta}(-\log|x-b|)\;, \tag{5.10}$$

still satisfies $\mathcal{N}(u_{D,\eta,b}\,, g_{\mathbb{R}^n}) = 0$. These functions present a singularity at $b\in\mathbb{R}^n$ and they are radial with respect to $b\in\mathbb{R}^n$. These new solutions $u_{D,\eta,b}(x)$ defined on $\mathbb{R}^n\setminus\{b\}$ correspond via (5.9) to the solutions $v_{D,\eta,b}(t,\theta)$ of the equation (3.1) defined on $\mathbb{R}\times\mathbb{S}^{n-1}\setminus\{(-\log|b|\,, b/|b|)\}$ by

$$v_{D,\eta,b}(t,\theta) \quad := \quad |\theta - be^{t}|^{-\frac{n-2k}{2k}} v_{D,\eta}(t-\log|\theta-be^{t}|)\;. \tag{5.11}$$

The last family of solutions comes in the following way. First observe that the function $t\mapsto \bar v_{D,\eta}(t) := v_{D,\eta}(-t)$ is still a solution to (3.1) on $\mathbb{R}\times\mathbb{S}^{n-1}$. This corresponds to the fact that on $\mathbb{R}\setminus\{0\}$ the Kelvin transform of $u_{D,\eta}$, namely the function

$$\bar u_{D,\eta}(|x|) \quad := \quad |x|^{-\frac{n-2k}{k}} u_{D,\eta}\big(|x/|x|^2|\big) \quad = \quad |x|^{-\frac{n-2k}{2k}} v_{D,\eta}\big(-\log|x/|x|^2|\big)\;, \tag{5.12}$$

satisfies $\mathcal{N}\left(\bar u_{D,\eta}\, g_{\mathbb{R}^n}\right) = 0$. Now we translate $\bar u_{D,\eta}$ by a vector $a\in\mathbb{R}^n$, obtaining an n-parameter family of functions $\bar u_{D,\eta,a}(x) := \bar u_{D,\eta}(|x-a|) = |x-a|^{-(n-2k)/2k} v_{D,\eta}(\log|x-a|)$ and finally we take the Kelvin transforms of the $\bar u_{D,\eta,a}$'s obtaining, for $a\in\mathbb{R}^n$ the new family of solutions on $\mathbb{R}^n\setminus\{0\}$

$$u_{D,\eta,a}(x) \quad := \quad \big|x-a|x|^2\big|^{-\frac{n-2k}{2k}} v_{D,\eta}\big(-2\log|x| + \log\big|x-a|x|^2\big|\;\big)\;. \tag{5.13}$$

These solutions are no longer radial and present a singularity at the origin. For $a\in\mathbb{R}^n$, they correspond on $\mathbb{R}\times S^{n-1}\setminus\{(\log|a|\,, a/|a|\,)\}$ to the solutions

$$v_{D,\eta,a}(t,\theta) \quad := \quad |\theta - ae^{-t}|^{-\frac{n-2k}{2k}} v_{D,\eta}(t+\log|\theta-ae^{-t}|)\;. \tag{5.14}$$

In the remaining part of this section we will use all these families of solutions to define some special elements in the kernel of the linearized operator around the Delaunay solutions $v_{D,\eta}$. First of all, we recall that if $\lambda\mapsto v_{D,\eta,\lambda}$ is a variation of $v_{D,\eta}$ such that for every admissible value of the parameter λ

$$\mathcal{N}\left(v_{D,\eta,\lambda}\,, g_{cyl}\right) \;=\; 0 \qquad \text{and} \qquad v_{D,\eta,0}(t) \;=\; v_{D,\eta}(t)\;,$$

then it is straightforward to see that

$$0 \;=\; \frac{\partial}{\partial\lambda}\Big|_{\lambda=0} \mathcal{N}\left(v_{D,\eta,\lambda}\,, g_{cyl}\right) \;=\; \mathbb{L}(v_{D,\eta}\,, g_{cyl})\; \frac{\partial}{\partial\lambda}\Big|_{\lambda=0} v_{D,\varepsilon,\lambda}\;,$$

where $\mathbb{L}(v_{D,\eta}\,, g_{cyl})$ represents the linearized operator around the Delaunay solution $v_{D,\eta}$. The functions $\partial_\lambda|_{\lambda=0}\, v_{D,\eta,\lambda}$ are the so called Jacobi fields and they clearly belong to the kernel of $\mathbb{L}(v_{D,\eta}\,, g_{cyl})$. Applying this reasoning to the family of solutions $\alpha\mapsto v_{D,\eta+\alpha}$ and $\tau\mapsto v_{D,\eta,\tau}$, it is natural to define the quantities

$$\Psi^{0,-}_\eta(t) \;:=\; \frac{\partial}{\partial\alpha}\Big|_{\alpha=0} v_{D,\eta+\alpha}(t) \qquad \text{and} \qquad \Psi^{0,+}_\eta(t) \;:=\; \frac{\partial}{\partial\tau}\Big|_{\tau=0} v_{D,\eta,\tau}(t) \;=\; \dot v_{D,\eta}(t)\;. \tag{5.15}$$

In analogy with that, we use the other two families $b\mapsto v_{D,\eta,b}$ and $a\mapsto v_{D,\eta,a}$ to define, for $j=1,\dots,n$, the Jacobi fields

$$\Psi^{j,-}_\eta(t,\theta) \quad := \quad \frac{\partial}{\partial b^j}\Big|_{b=0} v_{D,\eta,b}(t,\theta) \;=\; \left[\tfrac{n-2k}{2k} v_{D,\eta}(t) \,+\, \dot v_{D,\eta}(t)\right] e^{t}\,\cdot\,\phi_j(\theta)\;, \tag{5.16}$$

$$\Psi^{j,+}_\eta(t,\theta) \quad := \quad \frac{\partial}{\partial a^j}\Big|_{a=0} v_{D,\eta,a}(t,\theta) \;=\; \left[\tfrac{n-2k}{2k} v_{D,\eta}(t) \,-\, \dot v_{D,\eta}(t)\right] e^{-t}\,\cdot\,\phi_j(\theta)\;, \tag{5.17}$$

where the ϕ_j's are the n eigenfunction of the Laplacian on S^{n-1} with eigenvalue $n-1$, namely $-\Delta_\theta\phi_j = (n-1)\,\phi_j$, for $j=1,\dots,n$.

5.2 A linear problem on the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$

In this subsection we want to study the problem

$$\mathbb{L}(v_{D,\eta}, g_{cyl})\,[w] \quad = \quad f \quad \text{in } \mathbb{R} \times \mathbb{S}^{n-1} \,.$$

Following [16] we observe that the natural functional setting for this problem is given by weighted Hölder or Sobolev spaces. Both choices are essentially equivalent. However, in our argument we will use Hölder spaces. For a fixed weight parameter $\delta \in \mathbb{R}$ and $m \in \mathbb{N}$ we define the space

$$C^m_\delta(D_\eta) \quad := \quad \{u \in C^m(D_\eta) \ : \ \|u\|_{C^m_\delta} < +\infty\} \,,$$

where the weighted norm is defined by

$$\|u\|_{C^m_\delta(D_\eta)} \quad := \quad \sup_{\mathbb{R}\times\mathbb{S}^{n-1}} \textstyle\sum_{j=1}^m (\cosh t)^{-\delta}\, |\nabla^j u|\,(t,\theta) \,.$$

We point out that $|\cdot|$ and ∇ are respectively the norm and the Levi-Civita connection of the cylindrical metric g_{cyl}. In the same way we define for $\delta \in \mathbb{R}$, $m \in \mathbb{N}$ and $\beta \in (0,1)$ the weighted Hölder seminorm by

$$[\,u\,]_{C^{m,\beta}_\delta(D_\eta)} \quad := \quad \sup_{t\,\in\,\mathbb{R}}\ (\cosh t)^{-\delta}\,[\,u\,]_{C^{m,\beta}(\,(t-1,t+1)\times\mathbb{S}^{n-1}\,)} \,. \tag{5.18}$$

The weighted Hölder spaces are then given by

$$C^{m,\beta}_\delta(D_\eta) \quad := \quad \big\{u \in C^{m,\beta}(D_\eta) \ : \ \|u\|_{C^{m,\beta}_\delta} := \|u\|_{C^m_\delta} + [\,u\,]_{C^{m,\beta}_\delta} < +\infty\,\big\} \,. \tag{5.19}$$

Following the analysis in [18], one immediately find that

$$\mathbb{L}(v_{D,\eta}, g_{cyl}) \ : \ C^{2,\beta}_\delta(D_\eta) \quad \longrightarrow \quad C^{0,\beta}_\delta(D_\eta)$$

is Fredholm, provided $\delta \notin I_\eta$, where $I_\eta := \{\pm\delta_{j,\eta} \,:\, j \in \mathbb{N}\}$ is the set of the indicial roots of the operator $\mathbb{L}(v_{D,\eta}, g_{cyl})$ at both $+\infty$ and $-\infty$. In general the indicial roots (for a precise definition see [18]) do depend on the neck-size parameter η, but here it follows from the explicit knowledge of the Jacobi fields that $\delta_{0,\eta}$ and $\delta_{1,\eta}$ are independent of η. In particular the indicial root $\delta_{0,\eta} = 0$ is related to the Jacobi fields $\Psi^{0,-}_\eta$ and $\Psi^{0,+}_\eta$, which are respectively linearly growing and bounded in t, whereas the indicial root $\delta_{1,\eta} = 1$ has multiplicity n and is related to the Jacobi fields $\Psi^{j,-}_\eta$ and $\Psi^{j,+}_\eta$, $j = 1, \ldots, n$, which are respectively exponentially growing with rate e^t and exponentially decreasing with rate e^{-t}. We also point out that as a consequence of the inequality (5.6) it can be deduced that for every admissible value of the Delaunay parameter η, the indicial root $\delta_{2,\eta}$ verifies the inequality

$$\bar{\delta}(n,k) \ := \ \sqrt{\tfrac{2n(n-k)}{k(n-1)} + \big(\tfrac{n-2k}{2k}\big)^2} \quad \leq \quad \delta_{2,\eta} \,. \tag{5.20}$$

We are now in the position to prove the following

Lemma 5.2. *Let $1 < \delta$, then the operator*

$$\mathbb{L}(v_{D,\eta}, g_{cyl}) \ : \ C^{2,\beta}_{-\delta}(D_\eta) \quad \longrightarrow \quad C^{0,\beta}_{-\delta}(D_\eta)$$

is injective.

Proof. Since the functions which are involved in the conjugation (5.5) are bounded and positive, it is not restrictive to prove the result for the conjugate operator

$$\mathcal{L}_\eta \ : \ C^{2,\beta}_{-\delta}(D_\eta) \quad \longrightarrow \quad C^{0,\beta}_{-\delta}(D_\eta) \,.$$

Performing a standard separation of variables and projecting the equation along the eigenfunctions of Δ_θ, we note that for the low frequencies $j = 0, \ldots, n$ the space of the general solutions to the homogeneous

equation is spanned by the (conjugate) Jacobi fields $\Phi_\eta^{j,\pm} := h^{(k-1)/2}\Psi_\eta^{j,\pm}$. On the other hand, it is easy to check that, for $\delta > 1$, no one of these functions belongs to the weighted space $C^{2,\beta}_{-\delta}(D_\eta)$. Thus, it is sufficient to test the injectivity only for the high frequencies, $j \geq n+1$. Hence, suppose to have a function Φ such that

$$\mathcal{L}_\eta \,\Phi \;=\; 0 \qquad \text{and} \qquad \Phi\,(t,\theta) \;=\; \textstyle\sum_{j\geq n+1} \Phi^j(t)\,\phi_j\,(\theta)\;.$$

Since $\Phi \in C^{2,\beta}_{-\delta}(D_\eta)$, we have that

$$|\Phi|\,(t,\theta) \;\leq\; C\cdot(\cosh t)^{-\delta}$$

for some fixed $C > 0$. On the other hand, the maximum principle (which holds when $\mathcal{L}_\eta$ acts on the high frequencies, see [17]) gives

$$|\Phi|\,(T,\theta) \;\leq\; C\cdot(\cosh T)^{-\delta}$$

for every $T \in \mathbb{R}$. Letting $T \to +\infty$ we deduce that $\Phi \equiv 0$ and the proof is complete. $\square$

Using the fact that $\mathcal{L}_\eta$ is formally selfadjoint, it is standard to deduce (see [16]) that

$$\mathbb{L}(v_{D,\eta}, g_{cyl}) \;:\; C^{2,\beta}_{\delta}(D_\eta) \quad\longrightarrow\quad C^{0,\beta}_{\delta}(D_\eta)$$

is surjective for $\delta > 1$, $\delta \notin I_\eta$. Following [16] we are going to improve these first issue by showing that the surjectivity can be obtained on a smaller space. To do that it is convenient to set

$$W(D_{\eta,R}) \quad:=\quad \operatorname{span}\{\,\chi_R\,\Psi_\eta^{j,\pm} \;:\; j = 0,\dots,n\,\}\;,$$

where χ_R is a non decreasing smooth cut-off function which is identically equal to 1 for $t \geq R$ and which vanish for $t \leq R-1$. A simple adaptation of the ODE argument used in [16, Proposition 2.7], gives us the following

Lemma 5.3. *Let $1 < \delta < \bar{\delta}(n,k)$, then the operator*

$$\mathbb{L}(v_{D,\eta}, g_{cyl}) \;:\; C^{2,\beta}_{-\delta}(D_\eta) \oplus W(D_{\eta,R}) \quad\longrightarrow\quad C^{0,\beta}_{-\delta}(D_\eta)$$

is surjective.

5.3 A linear Dirichlet problem on the half cylinder $\mathbb{R}^+ \times \mathbb{S}^{n-1}$

In this subsection we study the Dirichlet problem

$$\left\{\begin{aligned} \mathbb{L}(v_{D,\eta}, g_{cyl})\,[w] \;&=\; f \qquad \text{in} \quad (R\,,+\infty)\times\mathbb{S}^{n-1}\;,\\ w \;&=\; 0 \qquad \text{on} \quad \{R\}\times\mathbb{S}^{n-1}\;, \end{aligned}\right. \tag{5.21}$$

for which will prove a well posedness result in the next Proposition 5.6. As it will be apparent from the proof, this result heavily relies on a proper choice of the value of R. Loosely speaking, the correct choice of R has to compensate the lack of maximum principle for the linear operator $\mathbb{L}(v_{D,\eta}, g_{cyl})$. The same kind of problem will show up also in the next subsection 5.4. For future convenience we set $D_{\eta,R} := (R\,,+\infty)\times\mathbb{S}^{n-1}$. Again, since h and $v_{D,\eta}$ are bounded and periodic we study without loss of generality the conjugate problem

$$\left\{\begin{aligned} \mathcal{L}_\eta\, z \;&=\; y \qquad \text{in} \quad D_{\eta,R}\;,\\ z \;&=\; 0 \qquad \text{on} \quad \partial D_{\eta,R}\;, \end{aligned}\right. \tag{5.22}$$

where $z = h^{(k-1)/2}w$ and $y = -C^{-1}_{n,k}v^{-1}_{D,\eta}h^{-(k-1)/2}f$. We consider now the usual eigenfunction decomposition

$$y\,(t,\theta) \;=\; \sum_{j=0}^{\infty} y^j(t)\;\phi_j(\theta) \qquad\text{and}\qquad z\,(t,\theta) \;=\; \sum_{j=1}^{\infty} z^j(t)\;\phi_j(\theta)\;,$$

where the ϕ_j's indicate the eigenfunctions of the Laplace-Beltrami operator on $(S^{n-1}, g_{S^{n-1}})$ which satisfy the identities $-\Delta_\theta \phi_j = \lambda_j \phi_j$, with $j \in \mathbb{N}$. We also recall that the spectrum of Δ_θ is given by $\{ m(n-2+m) \; : \; m \in \mathbb{N} \}$ and that in particular the first nonzero eigenvalue is $n-1$, with multiplicity n.

In the spirit of [15], it is convenient to treat separately the high frequencies, i.e., $j \geq n+1$, and the low frequencies, namely $j = 0, \ldots, n$. Basically, this distinction is motivated by the fact that, depending on the size of the λ_j, the quantity $a_\eta \lambda_j + p_\eta$ presents a change of sign and this has a clear influence on the analytical properties of our operators.

High frequencies: $j \geq n+1$. We consider the projection of z and y along the high frequencies

$$\bar{y}(t,\theta) \;:=\; \sum_{j=n+1}^{\infty} y^j(t)\,\phi_j(\theta) \qquad \text{and} \qquad \bar{z}(t,\theta) \;:=\; \sum_{j=n+1}^{\infty} z^j(t)\,\phi_j(\theta)\,,$$

and for $T > R$ we consider the projected and truncated linear Dirichlet problem

$$\begin{cases} \mathcal{L}_\eta\, \bar{z} \;=\; \bar{y} & \text{in} \quad D^T_{\eta,R}\,, \\ \quad\;\; \bar{z} \;=\; 0 & \text{on} \quad \partial D^T_{\eta,R}\,, \end{cases} \tag{5.23}$$

where $D^T_{\eta,R} := (R,T) \times \mathbb{S}^{n-1}$. In the high frequencies regime the linear problem (5.23) has a clear variational structure. Indeed it is easy to see that critical points of the Euler-Lagrange functional

$$E_T(\bar{z}) \;:=\; \int_R^T \int_{\mathbb{S}^{n-1}} \Big(|\,\partial_t \bar{z}\,|^2 \;+\; a_\eta\, |\,\nabla_\theta\, \bar{z}\,|^2_\theta \;+\; p_\eta\, \bar{z}^2 \;+\; 2\,\bar{y}\,\bar{z} \;\Big)\, dt\, d\theta \tag{5.24}$$

are weak solutions of (5.23) (here $d\theta$ represent the volume element of the round metric $g_{S^{n-1}}$ on the $(n-1)$-dimensional sphere). On the other hand, since $j \geq n+1$, we have by [17, Lemma 5.3] that $a_\eta\, \lambda_j + p_\eta > 0$ in $D_{\eta,R}$. This implies that the functional E_T is coercive on

$$\big[\, H^1_0\,(D^T_{\eta,R})\,\big]^\perp \;:=\; \left\{ u \,\in\, H^1_0(D^T_{\eta,R}) \;\middle|\; \int_{\mathbb{S}^{n-1}} u(\,\cdot\,,\theta)\;\phi_j(\theta)\;\, d\theta \;=\; 0\,, \quad j = 0, \ldots, n \right\}\,,$$

hence it is bounded from below. Furthermore it is easy to check that the functional E_T is weakly lower-semicontinuous on $\big[\, H^1_0\,(D^T_{\eta,R})\,\big]^\perp$. Thus, using the direct method of calculus of variations, we infer the existence of a minimizer $\bar{z}_T$ of E_T, which provides a (weak) solution of (5.23). The standard elliptic theory yields the expected regularity issues for $\bar{z}_T$ in terms of the regularity of $\bar{y}$.

Moreover, as a particular case of [17, Proposition 6.4], in the high frequencies regime, there holds the following

Lemma 5.4. *Let $|\,\delta\,| \,<\, \bar{\delta}(n,k)$, then there exists a positive constant $C = C(n,k,\delta) > 0$ such that if $\bar{z}_T \in C^{2,\beta}_{-\delta}(D^T_{\eta,R})$ and $y \in C^{0,\beta}_{-\delta}(D^T_{\eta,R})$ verify* (5.23)*, then we have*

$$\|\,\bar{z}_T\|_{C^{2,\beta}_{-\delta}(D^T_{\eta,R})} \;\;\leq\;\; C\;\, \|\,\bar{y}\,\|_{C^{0,\beta}_{-\delta}(D^T_{\eta,R})}\,, \tag{5.25}$$

for every $T > R$.

Using the fact that the estimate is independent of T, it is easy to obtain a solution $\bar{z}$ to (5.22) by letting $T \to +\infty$. Moreover it is clear that $\bar{z}$ verifies the estimate

$$\|\,\bar{z}\|_{C^{2,\beta}_{-\delta}(D_{\eta,R})} \;\;\leq\;\; C\;\, \|\,\bar{y}\,\|_{C^{0,\beta}_{-\delta}(D_{\eta,R})}\,, \tag{5.26}$$

with the same constant C as in Proposition 5.4.

Low frequencies: $j = 0, \ldots, n$. Here we start by considering the projection of our original problem (5.22) along the eigenfunction ϕ_0, obtaining

$$
\left\{
\begin{array}{rcl}
\mathcal{L}_\eta^{\,0}\, z^0 & = & y^0 \qquad \text{in} \quad (\, R \,,\, +\infty \,) \,, \\
z^0(R) & = & 0 \,.
\end{array}
\right.
\tag{5.27}
$$

where $\mathcal{L}^0_\eta := \partial^2_t - p_\eta$. As it is evident, in this case the potential has a wrong sign, thus we are forced to use a different approach in order to provide existence. We suppose that the right hand side is at least continuous and we extend it to the whole $\mathbb{R}$ (with a small abuse of notations, we still denote this extension by y^0). Next, following [15], we consider, for any $T > R$, the auxiliary backward Cauchy problem

$$
\left\{
\begin{array}{rcl}
\mathcal{L}_\eta^{\,0}\, z & = & y^{\,0} \qquad \text{in} \quad (\, -\infty \,,\, T \,) \,, \\
z\,(T) & = & 0 \,, \\
\dot z\,(T) & = & 0 \,.
\end{array}
\right.
\tag{5.28}
$$

Using the Cauchy-Lipschitz Theorem, we infer the existence of a unique solution z^0_T to (5.28). As we are going to show, the weighted norms of these solutions admit a bound which is uniform in T. This will allow us to produce a solution to the problem (5.27) with the wrong boundary data, just by taking the limit of the z^0_T's for $T \to +\infty$. As a final step we will correct these boundary data by adding a suitable multiple of the (conjugated) Jacobi field $\Phi^{0,+}_\eta := h^{(k-1)/2}\ \Psi^{0,+}_\eta = h^{(k-1)/2}\ \dot v_{D,\eta}$, which lies by definition in the kernel of $\mathcal{L}^0_\eta$.

Lemma 5.5. *Let $0 < \delta$ and $R = \tilde m\, T_\eta + \tilde r$ with $\tilde m \in \mathbb{N}$ and $\tilde r \in \mathbb{R}$ sufficiently small. Then, there exists a positive constant $C = C(n, k, \delta) > 0$ such that if $z^0_T \in C^{2,\beta}_{-\delta}(R, T)$ and $y^0 \in C^{0,\beta}_{-\delta}(R, T)$ verify (5.28), then we have*

$$
\|\, z^0_T \|_{C^{2,\beta}_{-\delta}(R,T)} \quad \leq \quad C \ \|\, y^0 \,\|_{C^{0,\beta}_{-\delta}((R,T)} \ ,
\tag{5.29}
$$

for every $T > R$.

Proof. We only prove the weighted C^0-estimate, since the weighted $C^{2,\beta}$-estimate will follow by standard scaling arguments. We want to establish the T-uniform bound

$$
\|\, z^0_T \|_{C^0_{-\delta}(R,T)} \quad \leq \quad C \ \|\, y^0 \,\|_{C^0_{-\delta}((R,T)} \ .
$$

We argue by contradiction. If the statement does not hold, then it is possible to find a sequence of triples $(T_i, z^0_{T_i}, y^0_i)$ such that

- $\mathcal{L}^0_\eta\, z^0_{T_i} = y^0_i \quad \text{in } (R, T_i) \qquad \text{and} \qquad z^0_{T_i}(T_i) \,=\, 0 \,=\, \dot z^0_{T_i}(T_i) \qquad \text{for every } i \in \mathbb{N}$,
- $\|\, z^0_{T_i} \,\|_{\mathcal{C}^{\,0}_{-\delta}(R,T_i)} \,=\, 1 \quad \text{for every } i \in \mathbb{N}$,
- $\|\, y^0_i \,\|_{\mathcal{C}^{\,0}_{-\delta}(R,T_i)} \longrightarrow\ 0 \quad \text{as } i \to +\infty$.

From the second point we infer the existence of a point $t_i \in (R, T_i)$ such that

$$
\sup_{t \in (R,T_i)} \ e^{\delta t}\ |\, z^0_{T_i} |\,(t) \quad = \quad e^{\delta t_i}\ |\, z^0_{T_i} |\,(t_i) \quad = \quad 1 \ .
$$

In fact on the half cylinder the weighting function $\cosh t$ can be replaced by e^t in the definition of the weighted norms. This yields an equivalent norm and simplify the computations of this subsection. It is now convenient to set, for every $i \in \mathbb{N}$,

$$
z_i\,(t) \ := \ e^{\delta t_i}\, z^0_{T_i}\,(t + t_i) \qquad \text{and} \qquad y_i\,(t) \ := \ e^{\delta t_i}\, y^{\,0}_i\,(t + t_i) \ ,
$$

where the point t varies now in $(\,-\log R - t_i \,,\, T_i - t_i\,)$, for every $i \in \mathbb{N}$. From these definitions it follows that

- $\mathcal{L}^0_\eta\, z_i = y_i$ in $(R-t_i, T_i-t_i)$ and $z_i(T_i-t_i) \,=\, 0 \,=\, \dot z_i^{\,0}(T_i-t_i)$ for every $i\in\mathbb{N}$,
- $\sup_{t\in(R-t_i\,,\,T_i-t_i)}\; e^{\delta t}\,|\,z_i|\,(t) \;=\; |\,z_i|\,(0) \;=\; 1$ for every $i\in\mathbb{N}$,
- $\sup_{t\in(-R-t_i\,,\,T_i-t_i)}\; e^{\delta t}\,|\,y_i|\,(t) \;\longrightarrow\; 0$ as $i\to+\infty$.

We are now ready to let $i\to+\infty$ and study the different limit situations in order to get a contradiction. As a first step we remark that (up to a subsequence) the intervals $(R-t_i\,,\,T_i-t_i)$ converge to an interval $(\beta^-\,,\,\beta^+)$ which is nonempty. In fact, since $R-t_i \,\le\, 0$ and $T_i-t_i \,\ge\, 0$, we have immediately that $\beta^-\in\mathbb{R}^-\cup\{-\infty\}$ and $\beta^+\in\mathbb{R}^+\cup\{+\infty\}$. Moreover, we claim that β^+ is strictly positive. In fact, if it would not be the case, then we would have that up to a subsequence $T_i-t_i\,\to\,0$. Since $|\,z_i|\,(0)\,=\,1$ and $z_i\,(T_i-t_i)\,=\,0\,=\,\dot z_i\,(T_i-t_i)$ for every $i\in\mathbb{N}$, the quantities $|\,\partial_t z_i\,|$'s must explode in the intervals $(\,T_i-t_i-1\,,\,T_i-t_i\,)$, as $i\to+\infty$. On the other hand, from the hypothesis on z_i and y_i it follows easily that

$$|\,\partial^2_t\, z_i|\,(t) \quad\le\quad C\,e^{-\delta\,(T_i-t_i)}\;,$$

on the interval $(\,T_i-t_i-1\,,\,T_i-t_i\,)$. Since we are supposing that $T_i-t_i\to 0$, this inequality tells us that the second derivatives of z_i are uniformly bounded as $i\to+\infty$. The fact that $\partial_t z_i\,(T_i-t_i)\,=\,0$ for every $i\in\mathbb{N}$ implies that the first derivatives of z_i also admit a uniform bound on $(\,T_i-t_i-1\,,\,T_i-t_i\,)$ as $i\to+\infty$, which is a contradiction. Hence we have that $(\beta^-,\,\beta^+)$ is always nonempty.

The equation satisfied by the z_i's implies that there exists a function w_∞ such that $w_i\,\to\,w_\infty$ in $\mathcal{C}^1_{loc}(\,\beta^-,\,\beta^+)$. In particular, the function z_∞ verifies the homogeneous equation

$$\mathcal{L}^0_\eta\, z_\infty \quad=\quad 0 \qquad \text{in}\ (\,\beta^-,\,\beta^+)\;, \tag{5.30}$$

in the sense of distributions. As a consequence z_∞ can be written as a linear combination of the Jacobi fields $\Phi^{0,-}_\eta$ and $\Phi^{0,+}_\eta$, namely, there exists $A,B\in\mathbb{R}$ such that

$$z_\infty \;=\; A\,\Phi^{0,-}_\eta + B\Phi^{0,+}_\eta\;.$$

Moreover, the hypothesis on $|\,z_i|\,(0)$ implies at once that $|\,z_\infty|\,(0)\;=\;1$. Thus z_∞ is non trivial. When $\beta^+<+\infty$, then the Cauchy data for the limit problem are given by $z_\infty\,(\beta^+)\;=\;0\;=\;\dot z_\infty\,(\beta^+)$, thus $z_\infty\equiv 0$ and we have a contradiction. If $\beta^+=+\infty$, then the decay prescription $|\,z_\infty|(t)\;\le\;e^{-\delta\,t}$ with $\delta>0$ implies that both the constants A and B must be zero, contradicting the non triviality of z_∞. $\square$

Since the estimate (5.29) is independent of the parameter $T\,>\,R$, we let $T\,\to\,+\infty$ and we obtain a function $\hat z^{\,0}$ which verifies the identity

$$\mathcal{L}^0_\eta\;\hat z^{\,0} \;=\; y^{\,0} \quad \text{in}\ (R,+\infty)$$

together with the T-uniform estimate

$$\|\,\hat z^0\|_{\mathcal{C}^{2,\beta}_{-\delta}(R\,,+\infty)} \quad\le\quad C\;\;\|\,y^0\|_{\mathcal{C}^{0,\beta}_{-\delta}(R\,,+\infty)}\;, \tag{5.31}$$

where $\delta>0$ and C is the same constant as in Lemma 5.5.

The next step amounts to correct the function $\hat z^{\,0}$ to a solution of the problem (5.27). This will be done by adding an element in the kernel of $\mathcal{L}^0_\eta$ to the function $\hat w_0$ in order to fulfill the homogeneous boundary condition at $t=R$. Here we decide to choose the (conjugated) Jacobi field $\Phi^{0,+}_\eta\;:=\;h^{(k-1)/2}\dot v_{D,\eta}$. We notice that this correction is no longer a function in $C^{2,\beta}_{-\delta}(R\,,+\infty)$, with $\delta>0$, since it is just bounded at $+\infty$. With these considerations, we are now ready to set

$$z^0(t) \quad:=\quad \hat z^{\,0}(t) \,-\, \frac{\hat z^{\,0}(R)}{\Phi^{0,+}_\eta(R)}\;\Phi^{0,+}_\eta(t)\;. \tag{5.32}$$

It is now immediate to check that this yields a solution to (5.27). We point out that the definition of w^0 makes sense since $R = \tilde{m}T_\eta + \tilde{r}$ and thus $\Phi^{0,+}_\eta(R) \neq 0$. From the definition of z^0 it follows at once that its component along the Jacobi field $\Phi^{0,+}_\eta$ is bounded by $(C/\Phi^{0,+}_\eta)(R)\, \| y^0 \|_{\mathcal{C}^{0,\beta}_{-\delta}(R\,,+\infty)}$ with C as in Proposition 5.5, hence for $\delta > 0$ the solution z^0 to (5.27) is unique in the space

$$C^{2,\beta}_{-\delta}(R\,,+\infty) \oplus \mathrm{span}\{ \Phi^{0,+}_\eta \} .$$

Now we are ready to treat the projection of (5.22) along the eigenfunction ϕ_j, with $j = 1, \dots, n$

$$\begin{cases} \mathcal{L}^{\,j}_\eta \, z^j \;=\; y^j & \text{in} \quad (\,R\,,+\infty\,)\,, \\ w^j\,(R) \;=\; 0\,, \end{cases} \tag{5.33}$$

where $\mathcal{L}^{\,j}_\eta := \partial^2_t - \lambda_j a_\eta - p_\eta$. Proceeding in the same manner as in the case $j = 0$ we deduce that for $\delta > 1$ there exists a unique solution $\hat{z}^j$ to this problem in the space

$$C^{2,\beta}_{-\delta}(R\,,+\infty) \oplus \mathrm{span}\{ \Phi^{j,+}_\eta \}$$

which can be written as

$$z^j \;=\; \hat{z}^j \;+\; \frac{\hat{z}^j(R)}{\big[\,\frac{n-2k}{2k} v_{D,\eta}(R) \,-\, \dot{v}_{D,\eta}(R)\,\big]}\; \Phi^{j,+}_\eta\,, \qquad j = 1, \dots n\,.$$

Note that this definition makes sense since, thanks to $h(t) > 0$ for any $t \in \mathbb{R}$ (see (5.1)), there holds that $\big[\frac{n-2k}{2k} v_{D,\eta}(R) \,-\, \dot{v}_{D,\eta}(R)\big] \neq 0$. Moreover we have as in the previous case that $\hat{z}^j$ verifies the estimate

$$\| \,\hat{z}^j \|_{C^{2,\beta}_{-\delta}(R\,,+\infty)} \quad\leq\quad C \;\, \| \,y^j \|_{C^{0,\beta}_{-\delta}(R\,,+\infty)}\,, \tag{5.34}$$

where now $\delta > 1$ and C is a positive constant only depending on n, k and δ and the component of z^j along the (conjugate) Jacobi fields $\Phi^{j,+}_\eta := h^{(k-1)/2}\Psi^{j,+}_\eta$ is bounded by $\big(\,C\,/\,[\,\frac{n-2k}{2k} v_{D,\eta}(R) \,-\, \dot{v}_{D,\eta}(R)\,]\,\big)\, \| \,y^j \|_{C^{0,\beta}_{-\delta}(R\,,+\infty)}$.

To summarize all the result of this subsection, we define the finite dimensional function space

$$\mathcal{W}^+(D_{\eta,R}) \;:=\; \mathrm{span}\big\{ \Psi^{j,+}_\eta \;:\; j = 0, \dots, n \big\}$$

and for a function $u = \sum_{j=0}^n a^+_j \, \Psi^{j,+}_\eta \in \mathcal{W}^+(D_{\eta,R})$ we simply set

$$\| \,u \,\|_{\mathcal{W}^+(D_{\eta,R})} \;:=\; \textstyle\sum_{j=0}^n |\, a^+_j \,|\,. \tag{5.35}$$

We thus have proved the following

Proposition 5.6. *Let $1 < \delta < \bar{\delta}(n,k)$ and $R = \tilde{m}T_\eta + \tilde{r}$ as above, then for every $f \in C^{0,\beta}_{-\delta}(D_{\eta,R})$ there exists a unique solution $w \in C^{2,\beta}_{-\delta}(D_{\eta,R}) \oplus \mathcal{W}^+(D_{\eta,R})$ to the problem*

$$\begin{cases} \mathbb{L}(v_{D,\eta}, g_{cyl})\,[w] \;=\; f & \textit{in} \quad (R\,,+\infty) \times \mathbb{S}^{n-1}\,, \\ \qquad\qquad\qquad\; w \;=\; 0 & \textit{on} \quad \{R\} \times \mathbb{S}^{n-1}\,. \end{cases}$$

Moreover we have that there exists a positive constant $C = C(n,k,\delta,\eta) > 0$ such that

$$\|w\|_{C^{2,\beta}_{-\delta}(D_{\eta,R}) \oplus \mathcal{W}^+(D_{\eta,R})} \;\;:=\;\; \|w\|_{C^{2,\beta}_{-\delta}(D_{\eta,R})} + \|w\|_{\mathcal{W}^+(D_{\eta,R})} \quad\leq\quad C \;\; \|f\|_{C^{0,\beta}_{-\delta}(D_{\eta,R})}\,.$$

The same analysis can be reproduced on a domain of the form $D_{\eta,-R} := (-\infty, -R) \times \mathbb{S}^{n-1}$, in order to solve the problem

$$\begin{cases} \mathbb{L}(v_{D,\eta}, g_{cyl})\,[w] \;=\; f & \text{in} \quad (-\infty,\, -R) \times \mathbb{S}^{n-1}\,, \\ \qquad\qquad\qquad\; w \;=\; 0 & \text{on} \quad \{-R\} \times \mathbb{S}^{n-1}\,, \end{cases} \tag{5.36}$$

where R is a real number of the form $R = \tilde{m}T_\eta + \tilde{r}$, as in Proposition 5.6. In this situation we use the finite dimensional function space

$$\mathcal{W}^-(D_{\eta,-R}) \;:=\; \mathrm{span}\left\{\Psi_\eta^{j,-} \;:\; j = 0, \ldots, n\right\}$$

for the corrections along the low frequencies. Obviously, for a function $u = \sum_{j=0}^n a_j^- \, \Psi_\eta^{j,-} \in \mathcal{W}^+(D_{\eta,-R})$ the norm can be defined by

$$\|\, u \,\|_{\mathcal{W}^-(D_{\eta,-R})} \;:=\; \textstyle\sum_{j=0}^n |\, a_j^- \,| \;.$$

In analogy with the previous proposition, it is straightforward to obtain the following

Proposition 5.7. *Let $1 < \delta < \bar{\delta}(n,k)$ and $R = \tilde{m}\,T_\eta + \tilde{r}$ as above, then for every $f \in C^{0,\beta}_{-\delta}(D_{\eta,-R})$ there exists a unique solution $w \in C^{2,\beta}_{-\delta}(D_{\eta,-R}) \oplus \mathcal{W}^-(D_{\eta,-R})$ to the problem (5.36). Moreover we have that there exists a positive constant $C = C(n,k,\delta,\eta) > 0$ such that*

$$\|w\|_{C^{2,\beta}_{-\delta}(D_{\eta,-R}) \oplus \mathcal{W}^-(D_{\eta,-R})} \;:=\; \|w\|_{C^{2,\beta}_{-\delta}(D_{\eta,-R})} + \|w\|_{\mathcal{W}^-(D_{\eta,-R})} \;\;\leq\;\; C \;\, \|f\|_{C^{0,\beta}_{-\delta}(D_{\eta,-R})} \;.$$

To conclude we observe that as a consequence of the analysis of this subsection we can also solve the same problems with nonzero boundary condition, namely for every $1 < \delta < \bar{\delta}(n,k)$, every $f \in C^{0,\beta}_{-\delta}(D_\eta)$ and every boundary datum $v^\pm \in C^{2,\beta}(\partial D_{\eta,\pm R})$, there exists a unique function $w^\pm \in C^{2,\beta}_{-\delta}(D_{\eta,\pm R}) \oplus \mathcal{W}^\pm(D_{\eta,\pm R})$ verifying

$$\begin{cases} \mathbb{L}(v_{D,\eta}, g_{cyl})\,[\,w^\pm] \;=\; f & \text{in} \quad D_{\eta,\pm R} \;, \\ \qquad\qquad\qquad w^\pm \;=\; v^\pm & \text{on} \quad \partial D_{\eta,\pm R} \;, \end{cases}$$

together with the estimate

$$\|\, w^\pm\|_{C^{2,\beta}_{-\delta}(D_{\eta,\pm R}) \oplus \mathcal{W}^-\pm(D_{\eta,\pm R})} \;\;\leq\;\; C \;\left[\; \|f\|_{C^{0,\beta}_{-\delta}(D_{\eta,\pm R})} \;+\; \|\, v^\pm\|_{C^{2,\beta}(\partial D_{\eta,\pm R})} \,\right] \,,$$

for some positive constant $C = C(n,k,\delta,\eta,R) > 0$. We thus have proved

Proposition 5.8. *Let $1 < \delta < \bar{\delta}$ and $R = \tilde{m}\,T_\eta + \tilde{r}$ as above, then the operator*

$$\mathbb{L}(v_{D,\eta}, g_{cyl}) \,\otimes\, \partial^\pm \;:\; C^{2,\beta}_{-\delta}(D_{\eta,\pm R}) \,\oplus\, \mathcal{W}^\pm(D_{\eta,\pm R}) \;\;\longrightarrow\;\; C^{0,\beta}_{-\delta}(D_{\eta,\pm R}) \,\times\, C^{2,\beta}(\partial D_{\eta,\pm R})$$

is an isomorphism, where $\partial^\pm : w \,\mapsto\, w_{|\partial D_{\eta,\pm R}}$.

5.4 A linear Dirichlet problem on a finite cylinder $(-R,R) \times \mathbb{S}^{n-1}$

In this subsection we study the Dirichlet problem

$$\begin{cases} \mathbb{L}(v_{D,\eta}, g_{cyl})\,[w] \;=\; f & \text{in} \quad (-R\,,R) \times \mathbb{S}^{n-1} \;, \\ \qquad\qquad\qquad w \;=\; 0 & \text{on} \quad \{R,-R\} \times \mathbb{S}^{n-1} \;, \end{cases} \tag{5.37}$$

for which we are going to prove the following

Proposition 5.9. *Let $R = \tilde{m}\,T_\eta + \tilde{r}$ with $\tilde{m} \in \mathbb{N}$ sufficiently large and $\tilde{r} \in \mathbb{R}$ sufficiently small. Then, for any $f \in C^{0,\beta}((-R,R) \times \mathbb{S}^{n-1})$ there exists a unique solution $w \in C^{2,\beta}((-R,R) \times \mathbb{S}^{n-1})$ to (5.37). Moreover, there exists a positive constant $C = C(n,k)$ such that*

$$\|w\|_{C^{2,\beta}((-R,R)\times\mathbb{S}^{n-1})} \;\leq\; C \;\, \|f\|_{C^{0,\beta}((-R,R)\times\mathbb{S}^{n-1})}. \tag{5.38}$$

Proof. Thanks to the compactness of the domain, we use the Fredholm alternative to prove the well posedness of (5.37). Thus, the existence and uniqueness of solutions to (5.37) follows from the fact that the homogeneous problem

$$\begin{cases} \mathbb{L}(v_{D,\eta}, g_{cyl})\,[w] \;=\; 0 & \text{in} \quad (-R\,,R)\times\mathbb{S}^{n-1}\;, \\ \qquad\qquad\quad w \;=\; 0 & \text{on} \quad \{\pm R\}\times\mathbb{S}^{n-1}\;, \end{cases}$$

only admits the trivial solution $w\equiv 0$. This is equivalent to say that $z\equiv 0$ is the unique solution of the conjugate problem

$$\begin{cases} \mathcal{L}_\eta\, z \;=\; 0 & \text{in} \quad (-R,R)\times\mathbb{S}^{n-1}\;, \\ \quad\; z \;=\; 0 & \text{on} \quad \{\pm R\}\times\mathbb{S}^{n-1}\;, \end{cases} \tag{5.39}$$

where $z := h^{(k-1)/2}w$. To show that $z=0$ uniquely solves (5.39) we project the equation along the eigenfunctions ϕ_j of the Laplace Beltrami operator on the sphere $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$. Then, as we already did in the proof of Proposition 5.6, we treat separately the high frequencies ($j\geq n+1$) and the low frequencies ($j=0,\ldots,n$). Thus, we decompose z as

$$z\,(t,\theta) \;=\; \sum_{j=0}^{n} z^j(t)\,\phi_j(\theta) \,+\, \sum_{j=n+1}^{\infty} z^j(t)\,\phi_j(\theta)\;. \tag{5.40}$$

Now, since in the high frequencies regime the (weak) solutions to (5.39) can be obtained as critical points of the coercive and weakly lower semicontinuous energy defined in (5.24), it turns out that the second sum in (5.40) is identically equal to zero.

To obtain the same result for the low frequencies, we start with the case $j=1,\ldots n$ and we note that the Fourier coefficients z^j can be written as a linear combination of $\Phi^{j,+}_\eta\,\phi_j(\theta)$ and $\Phi^{j,-}_\eta\,\phi_j(\theta)$, where the conjugated Jacobi fields $\Phi^{j,\pm}_\eta$ are defined as $\Phi^{j,\pm}_\eta := h^{(k-1)/2}\Psi^{j,\pm}_\eta$. Hence, there exist real numbers A_j and B_j such that

$$z^j(t) \;=\; h^{(k-1)/2}(t)\,\left\{A_j\left[\tfrac{n-2k}{2k}v_{D,\eta}(t)\,+\,\dot v_{D,\eta}(t)\right]e^t + B_j\left[\tfrac{n-2k}{2k}v_{D,\eta}(t)\,-\,\dot v_{D,\eta}(t)\right]e^{-t}\right\}.$$

We are going to show that the homogeneous Dirichlet boundary conditions together with our choice of R imply that all the z^j must vanish for every $j=1,\ldots,n$. From the null boundary condition we deduce that

$$\begin{cases} (A_j+B_j)\,\left[\tfrac{n-2k}{2k}\,+\,\dfrac{\dot v_{D,\eta}(R)}{v_{D,\eta}(R)}\tanh(R)\right] \;=\; 0, \\ (A_j-B_j)\,\left[\tfrac{n-2k}{2k}\tanh(R)\,+\,\dfrac{\dot v_{D,\eta}(R)}{v_{D,\eta}(R)}\right] \;=\; 0, \end{cases}$$

where we have used the fact that $v_{D,\eta}(\cdot)$ is an even function. Now, using the fact that R is of the form $R=\tilde m T_\eta+\tilde r$, it is sufficient to choose $\tilde m$ large enough and $\tilde r$ small enough in order to insure that

$$\begin{cases} \tfrac{n-2k}{2k}\,+\,\dfrac{\dot v_{D,\eta}(R)}{v_{D,\eta}(R)}\tanh(R) \;\neq\; 0, \\ \tfrac{n-2k}{2k}\tanh(R)\,+\,\dfrac{\dot v_{D,\eta}(R)}{v_{D,\eta}(R)} \;\neq\; 0. \end{cases}$$

Hence, $A_j=0=B_j$, for every $j=1,\ldots,n$.

It remains to prove that also $z^0\equiv 0$. Now, z^0 has this form

$$z^0(t) \;= h^{(k-1)/2}(t)\;\left\{A_0\,\Psi^{0,-}_\eta(t)+\,B_0\,\Psi^{0,+}_\eta(t)\right\}\;,$$

for suitable A_0 and B_0 in $\mathbb{R}$. From the homogeneous boundary conditions combined with the fact that $\Psi^{0,+}_\eta$ is odd and $\Psi^{0,-}_\eta$ is even, we deduce at once that if R is chosen as in the statement both A_0 and B_0 must vanish.

Hence, the problem (5.37) is well posed. On the other hand the a priori estimate (5.38) is a direct consequence of the standard elliptic regularity theory. $\square$

6 Global linear analysis

The aim of this section is to provide existence, uniqueness and *a priori* estimates for solutions to the linear problem

$$\mathbb{L}(u_\varepsilon, \bar g)\,[w] \;=\; f \qquad \text{in } M_\varepsilon. \tag{6.1}$$

For sake of simplicity we just consider the case where M_ε is the connected sum of two Delaunay type solution $M_\varepsilon = D_{\eta_1} \sharp_\varepsilon D_{\eta_2}$. All the arguments can be trivially adapted to the general case $M_\varepsilon = D_{\eta_1} \sharp_\varepsilon \ldots \sharp_\varepsilon D_{\eta_N}$.

To introduce the correct functional framework for the global linear analysis on M_ε we define

$$\|u\|_{C^{m,\beta}_{\delta,\gamma}(M_\varepsilon)} \;:=\; \|u\|_{C^{m,\beta}_{\delta}(D_{\eta_1}\setminus B(p_1,1))} + \|u\|_{C^{m,\beta}_{\delta}(D_{\eta_2}\setminus B(p_2,1))} + \|u\|_{C^{m,\beta}_{\gamma}(N_\varepsilon)}\,, \tag{6.2}$$

where the first two norms are defined as in Subsection 5.2 and the norm in the neck region N_ε is defined by

$$\begin{aligned} \|u\|_{C^{m,\beta}_{\gamma}(N_\varepsilon)} \;:=\; & \sup_{N_\varepsilon} \; \textstyle\sum_{j=1}^m (\varepsilon \cosh t)^\gamma \, |\nabla^j u|\,(t,\theta) \\ & + \sup_{t\,\in\,(\log\varepsilon, -\log\varepsilon)} (\varepsilon\cosh t)^\gamma \,[\,u\,]_{C^{m,\beta}(\,(t-1,t+1)\times\mathbb{S}^{n-1}\,)}\,. \end{aligned} \tag{6.3}$$

Note that, again, $|\cdot|$ and ∇ are respectively the norm and the Levi-Civita connection of the cylindrical metric g_{cyl} whereas $[\cdot]$ stands for the usual Hölder seminorm. As a consequence, we introduce the following weighted Hölder space

$$C^{m,\beta}_{\delta,\gamma}(M_\varepsilon) \;:=\; \{u \in C^{m,\beta}_{loc}(M_\varepsilon) \;:\; \|u\|_{C^{m,\beta}_{\delta,\gamma}(M_\varepsilon)} < +\infty\}\,.$$

6.1 Global and uniform *a priori* estimates on M_ε

We recover from [4] the following removable singularities result

Lemma 6.1 (Removable singularities)**.** *Let $g = (1+b)^{\frac{4k}{n-2k}} g_{\mathbb{R}^n}$ be a conformally flat metric defined on a geodesic ball $B(p,1)$ verifying the equation*

$$\sigma_k(g^{-1}A_g) \;=\; 2^{-k}\tbinom{n}{k}\,.$$

Suppose $\bar w$ satisfies in the sense of distributions

$$\mathbb{L}(1+b, g_{\mathbb{R}^n})[\bar w] \;=\; 0 \quad on \quad B^*(p,1)$$

with $|\bar w(q)| \le C|dist_g(q,p)|^{-\mu}$ for any $q \in B^*(p,1)$ for some positive constant $C>0$ and for some weight parameter $0<\mu<n-2$. Then $\bar w$ is a bounded smooth function on $B(p,1)$ and satisfies the equation above on the entire ball.

We are now in the position to prove the following ε-uniform *a priori* estimate for solutions to (6.1).

Proposition 6.2. *Suppose that $1 < \delta < \bar{\delta}\,(n,k)$, $0 < \gamma < (n-2k)/k$ and let $w \in C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)$ and $f \in C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)$ be two functions satisfying*

$$\mathbb{L}(u_\varepsilon, \bar{g})\,[w] \;=\; f \qquad \text{in } M_\varepsilon\,.$$

Then there exist $C = C(n,k,\beta,\gamma,\delta) > 0$ and $\varepsilon_0 = \varepsilon_0(n,k,\gamma,\delta)$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, we have

$$\|w\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}} \;\leq\; C\,\|f\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}}\,. \tag{6.4}$$

Proof. Before starting the proof, we advise the reader that we will prove (6.4) using a different, albeit equivalent, norm. With a little abuse of notation, we introduce the norm

$$\|u\|_{C^{m,\beta}_{-\delta,\gamma}(M_\varepsilon)} \;:=\; \max\Big\{\|u\|_{C^{m,\beta}_{-\delta}(D_{\eta_1}\setminus B(p_1,1))},\, \|u\|_{C^{m,\beta}_{-\delta}(D_{\eta_2}\setminus B(p_2,1))},\, \|u\|_{C^{m,\beta}_{\gamma}(N_\varepsilon)}\Big\}\,,$$

which is clearly equivalent to (6.2). We will just provide the uniform weighted C^0–bound

$$\|w\|_{C^{0}_{-\delta,\gamma-\frac{n-2k}{2k}}} \;\leq\; C\,\|f\|_{C^{0}_{-\delta,\gamma-(n-2k)}}\,,$$

since the uniform weighted $C^{2,\beta}$–bound will follows easily from standard scaling argument, see [18]. To prove the above estimate we argue by contradiction. Suppose that there exists a sequence $(\varepsilon_i, w_i, f_i)$, $i \in \mathbb{N}$, such that

- $\varepsilon_i \longrightarrow 0$, as $i \to +\infty$,
- $\|w_i\|_{C^{0}_{-\delta,\gamma-\frac{n-2k}{2k}}} \equiv 1$, $i \in \mathbb{N}$,
- $\|f_i\|_{C^{0}_{-\delta,\gamma-(n-2k)}} \longrightarrow 0$, as $i \to +\infty$

and

$$\mathbb{L}(u_{\varepsilon_i}, \bar{g})\,[w_i] \;=\; f_i \qquad \text{in } M_\varepsilon.$$

Now, up to a not relabelled subsequence of i, one has to face with the following two cases:

1. $\|w_i\|_{C^0_{-\delta}(D_{\eta_j}\setminus B(p_j,1))} = 1$, for any $i \in \mathbb{N}$ and for $j = 1$ or 2.

2. $\|w_i\|_{C^0_\gamma(N_\varepsilon)} = 1$ for any $i \in \mathbb{N}$.

The second case has been fully analysed in [4, Proposition 4.4, case 2. and case 3.] to which we refer for the details. Concerning the first one, we note that there is no loss of generality in restricting the analysis only to D_{η_1} (recall that on D_{η_1} we use coordinates r_1 and θ, according to Section 4). Secondly, it is natural to split the case 1 into two subcases. The first subcase appears when there exists $M > 0$ and a subsequence of points $q_i = (r_i, \theta_i)$'s such that $r_i \in [-M, M] \times \mathbb{S}^{n-1}$ and $(\cosh r_i)^\delta |w_i|(r_i,\theta_i) \geq 1/2$. The second subcase is when the points q_i's leave every compact set of D_{η_1}. However, it follows from (5.37) that this second subcase can always be reduced to the first one. The assumption $q_i \in [-M, M\,] \times \mathbb{S}^{n-1}$ implies that, up to a not relabelled subsequence, there holds that $q_i \to q_\infty$, $w_i \to w_\infty$ in $C^2_{loc}(D_{\eta_1}\setminus\{p_1\})$ and $f_i \to 0$ in $C^0_{loc}(D_{\eta_1}\setminus\{p_1\})$. Thus, the function w_∞ is nontrivial, since $|w_\infty|(q_\infty) \geq (\cosh M)^{-\delta}/2$, and solves in the sense of distributions the limit problem

$$\mathbb{L}(1, g_1)\;[u_1^{-1}\,w_\infty] \;=\; 0 \qquad \text{in } D_{\eta_1}\setminus\{p_1\} \qquad \text{with} \qquad \|\,w_\infty\|_{C^0_{-\delta}(D_{\eta_1}\setminus B(p_1,1))} \;=\; 1\,,$$

where u_1 is the function used in the construction of the approximate solutions (see Section 4), which we have set to be equal to 1 in $D_{\eta_1}\setminus B(p_1,1)$. If we show that the limit problem is solved by $u_1^{-1}\,w_\infty$ on

the whole D_{η_1}, then, using Lemma 5.2 (which is in force thanks to the fact that $1 < \delta$), we will reach the desired contradiction. To remove the singularity at p_1, we observe that on $B(p_1, 1)$ we can always write

$$g_1 = (1 + b_1)^{\frac{4k}{n-2k}} g_{\mathbb{R}^n},$$

with $b_1(q) = \mathcal{O}\big(|dist_{g_1}(q, p_1)|^2\big)$. Hence, thanks to the conformal equivariance property (2.4), we have that the limit equation can be rewritten as

$$\mathbb{L}\left((1 + b_1)\, , g_{\mathbb{R}^n} \right) \left[(1 + b_1) u_1^{-1} w_\infty \right] \;=\; 0 \quad \text{on} \quad B(p_1, 1) \setminus \{p_1\} \; .$$

Recalling that g_1 solves the σ_k–Yamabe equation and that

$$|(1 + b_1) u_1^{-1} w_\infty|(q) \;\leq\; C \; |dist_{g_1}(q, p_1)|^{-\gamma} \; ,$$

we can apply Lemma 6.1 to obtain that $u_1^{-1} w_\infty$ extends through p_1 to a nontrivial smooth solution of

$$\mathbb{L}\,(1\, , g_1)\; [u_1^{-1} w_\infty] \;=\; 0 \quad \text{in} \quad D_{\eta_1} \; .$$

This completes the proof. $\square$

As a consequence of the *a priori* estimates, we obtain the following

Corollary 6.3. *Suppose that $1 < \delta < \bar{\delta}\,(n, k)$ and $0 < \gamma < (n - 2k)/k$, then there exists a positive real number $\varepsilon_0(n, k, \gamma, \delta) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ the operator*

$$\mathbb{L}(u_\varepsilon, \bar{g}) \;:\; C^{2,\beta}_{-\delta, \gamma - \frac{n-2k}{2k}}(M_\varepsilon) \quad \longrightarrow \quad C^{0,\beta}_{-\delta, \gamma-(n-2k)}(M_\varepsilon)$$

is injective.

The next task is to provide surjectivity for $\mathbb{L}(u_\varepsilon, \bar{g})$. Unfortunately, with this choice of the weight parameter, which will turn out to be suitable for the nonliner analysis, the surjectivity cannot be recovered. The duality theory would suggest the use of the spaces with weight parameter δ insted of $-\delta$, but as it is remarked in [16], these spaces are definitely too large. In particular they contain functions which may grow too fast at $\pm\infty$ and even worst which are not *integrable*, in the sense which is made precise below. To overcome this difficulty, one can take advantage of the Fredholm character of the operator (which is actually the case, when δ is not an indicial root) by considering a finite dimensional extension of the domain, the so called *deficiency space*. Of course, there are several different options for the choice of such a space (for example a different approach is contained in [16]). We start by introducing the following spaces

$$W(D_{\eta_1, R_1}) := \text{span}\,\{\, \chi_{R_1'}\, \Psi^{j,\pm}_{\eta_1} \;:\; j = 0, \ldots, n \,\} \quad \text{and} \quad W(D_{\eta_1, -R_1}) := \text{span}\,\{\, \chi_{-R_1'}\, \Psi^{j,\pm}_{\eta_1} \;:\; j = 0, \ldots, n \,\}$$
$$W(D_{\eta_2, R_2}) := \text{span}\,\{\, \chi_{R_2'}\, \Psi^{j,\pm}_{\eta_2} \;:\; j = 0, \ldots, n \,\} \quad \text{and} \quad W(D_{\eta_2, -R_2}) := \text{span}\,\{\, \chi_{-R_2'}\, \Psi^{j,\pm}_{\eta_2} \;:\; j = 0, \ldots, n \,\}$$
$$\mathcal{W}^+(D_{\eta_1, +R_1}) := \text{span}\,\{\, \chi_{+R_1'}\, \Psi^{j,+}_{\eta_1} \;:\; j = 0, \ldots, n \,\} \quad \text{and} \quad \mathcal{W}^+(D_{\eta_2, +R_2}) := \text{span}\,\{\, \chi_{+R_2'}\, \Psi^{j,+}_{\eta_2} \;:\; j = 0, \ldots, n \,\}$$
$$\mathcal{W}^-(D_{\eta_1, -R_1}) := \text{span}\,\{\, \chi_{-R_1'}\, \Psi^{j,-}_{\eta_1} \;:\; j = 0, \ldots, n \,\} \quad \text{and} \quad \mathcal{W}^-(D_{\eta_2, -R_2}) := \text{span}\,\{\, \chi_{-R_2'}\, \Psi^{j,-}_{\eta_2} \;:\; j = 0, \ldots, n \,\}$$

where $\chi_{R_1'}$ is a non decreasing smooth cut-off function defined on D_{η_1} which is identically equal to 1 for $t \geq R_1'$ and which vanish for $t \leq R_1' - 1$ (with $R_1' - 1 > R_1$. The cut-off function $\chi_{-R_1'}$ is defined by the relationship $\chi_{-R_1'}(r) = \chi_{R_1'}(-r)$, and $\chi_{R_2'}$ and $\chi_{-R_2'}$ are defined on D_{η_2} in an analogous fashion. Moreover, the parameters R_1' and R_2' are chosen in such a way that $R_1' - 1 > R_1$ and $R_2' - 1 > R_2$, in order to apply the analysis of the previous section.

We observe that all the functions in these spaces are *integrable* at $\pm\infty$ in the sense that they are linear combinations of infinitesimal generators of families of conformal variations of the original Delaunay type solutions v_{D,η_1} and v_{D,η_2}. This fact has an important geometrical meaning which will be made clear later

and which will be used in the nonlinear framework to justify the choice of a correction w with components lying in these spaces. To continue, we define the full deficiency space $\overline{W}(M_\varepsilon)$ by

$$\overline{W}(M_\varepsilon) \quad := \quad W(D_{\eta_1,R_1}) \oplus W(D_{\eta_1,-R_1}) \oplus W(D_{\eta_2,R_2}) \oplus W(D_{\eta_2,-R_2})\ .$$

We notice that since all of these spaces are finite dimensional, we can choose any norm on them. To be definite we always consider the norm given by the sum of the absolute value of the components.

The importance of these spaces for the linear analysis is clarified by the following Proposition, which can be deduced combining Corollary 6.3 with [18, Theroem 12.4.2],

Proposition 6.4. *Suppose that $1 < \delta < \bar{\delta}\,(n,k)$ and $0 < \gamma < (n-2k)/k$, then there exists a positive real number $\varepsilon_0(n,k,\gamma,\delta) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ the operator*

$$\mathbb{L}(u_\varepsilon, \bar{g})\ :\ C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\,\oplus\,\overline{W}(M_\varepsilon) \quad\longrightarrow\quad C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)$$

is surjective and

$$\dim\,\mathrm{Ker}\,\mathbb{L}\,(u_\varepsilon, \bar{g}) \;=\; \tfrac{1}{2}\,\dim \overline{W}(M_\varepsilon) \;=\; 4\,(n+1)\ .$$

Our *deficiency space* is defined by

$$\mathcal{W}(M_\varepsilon) \quad := \quad \mathcal{W}^+(D_{\eta_1,R_1}) \oplus \mathcal{W}^-(D_{\eta_1,-R_1}) \oplus \mathcal{W}^+(D_{\eta_2,R_2}) \oplus \mathcal{W}^-(D_{\eta_2,-R_2})\ . \tag{6.5}$$

Incidentally we note that in [16], due to the use of a different functional framework, the *deficiency space* is chosen to be

$$W(M_\varepsilon) \quad := \quad W(D_{\eta_1,R_1}) \oplus W(D_{\eta_2,R_2})\ .$$

As expected, $\dim W(M_\varepsilon) \,=\, 4\,(n+1) \,=\, \dim \mathcal{W}(M_\varepsilon)$.

The remaining part of this section will be devoted to prove the core of our linear analysis, namely the following

Proposition 6.5. *Suppose that $1 < \delta < \bar{\delta}\,(n,k)$ and $0 < \gamma < (n-2k)/k$, then there exists a positive real number $\varepsilon_0(n,k,\gamma,\delta) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ the operator*

$$\mathbb{L}(u_\varepsilon, \bar{g})\ :\ C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\,\oplus\,\mathcal{W}(M_\varepsilon) \quad\longrightarrow\quad C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)$$

is an isomorphism. Moreover the following ε-uniform a priori estimate is satisfied

$$\|w\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)} \;+\; \|w\|_{\mathcal{W}(M_\varepsilon)} \quad\leq\quad C\ \|\,\mathbb{L}\,(u_\varepsilon,\bar{g})\,[w]\,\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)}\ , \tag{6.6}$$

where the positive constant C only depends on n, k, β, γ and δ.

Proof. First of all, Proposition 6.4 furnishes for any $f \in C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)$ the existence of a solution u to (6.1) in the class $C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\,\oplus\,\overline{W}(M_\varepsilon)$. We canonically decompose $\overline{W}$ as $\overline{W} \,=\, \mathcal{W} \oplus \mathcal{W}^\perp$, where $\mathcal{W}^\perp$ represent the orthogonal complement of $\mathcal{W}$ in $\overline{W}$. Consequentely, u admits the decomposition

$$u \;=\; \hat{u} + u^\top + u^\perp,$$

where $\hat{u} + u^\top \;\in\; C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\,\oplus\,\mathcal{W}(M_\varepsilon)$, whereas $u^\perp \;\in\; \mathcal{W}^\perp$. The aim is thus to find a correction $z \in C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\,\oplus\,\mathcal{W}(M_\varepsilon)$ such that, the function w defined by

$$w \;:=\; \hat{u} + u^\top + z,$$

is a solution to (6.1). It is clear that, as soon as we are able to produce such a correction, then the surjectivity in the smaller space $C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)$ will be achieved. Using the linearity of the problem and the fact that u is already a solution, one can easily deduce that the function z must satisfy

$$\mathbb{L}(u_\varepsilon,\bar g)\,[z] \;=\; \mathbb{L}(u_\varepsilon,\bar g)\,[u^\perp] \qquad \text{in } M_\varepsilon.$$

A remarkable feature of problem above is that the right hand side is supported by construction only in $D_{\eta_1,\pm R_1}\cup D_{\eta_2,\pm R_2}$. More precisely, recalling that $R_1'>R_1$ and $R_2'>R_2$, we have that $\mathbb{L}(u_\varepsilon,\bar g)\,[u^\perp]$ is actually supported in the four annuli $[R_1'-1,R_1']\times\mathbb{S}^{n-1}$, $[-R_1',-R_1'+1]\times\mathbb{S}^{n-1}$, $[R_2'-1,R_2']\times\mathbb{S}^{n-1}$ and $[-R_2',-R_2'+1]\times\mathbb{S}^{n-1}$. As a consequence, we are led to solve the following kind of problem

$$\left\{\begin{array}{rcll} \mathbb{L}(u_\varepsilon,\bar g)\,[\,v_{1,+}] &=& \mathbb{L}(u_\varepsilon,\bar g)\,[\,u^\perp] \text{ in } & D_{\eta_1,R_1}\,, \\ v_{1,+} &=& 0 \qquad\qquad \text{on} & \partial D_{\eta_1,R_1}, \end{array}\right. \tag{6.7}$$

with $v_{1,+}\in C^{2,\beta}_{-\delta}(D_{\eta_1,R_1})\oplus\mathcal{W}^+(D_{\eta_1,R})$. Analogous problems should be posed, respectively, on $D_{\eta_1,-R_1}$, D_{η_2,R_2}, $D_{\eta_2,-R_2}$, leading to the construction of $v_{1,-},v_{2,+},v_{2,-}$. Problem (6.7) is, modulo the use of the conformal equivariance property (2.4), of the same kind of the Dirichlet problem (5.21). Thus, Proposition 5.6 applies giving the well posedness of (6.7). Once we have obtained these local solutions, we define the candidate correction z as

$$z \;:=\; \left\{\begin{array}{cll} v_{1,+}+\bar v_{1,+} & \text{in} & D_{\eta_1,R_1} \\ v_{1,-}+\bar v_{1,-} & \text{in} & D_{\eta_1,-R_1} \\ v_C & \text{in} & C_\varepsilon := M_\varepsilon\setminus(D_{\eta_1,R_1}\cup D_{\eta_1,-R_1}\cup D_{\eta_2,R_2}\cup D_{\eta_2,-R_2}) \\ v_{2,+}+\bar v_{2,+} & \text{in} & D_{\eta_2,R_2} \\ v_{2,-}+\bar v_{2,-} & \text{in} & D_{\eta_2,-R_2}, \end{array}\right. \tag{6.8}$$

where $\bar v_{1,+},\bar v_{1,-},\bar v_{2,+},\bar v_{2,-},v_C$ are the solutions of the following problems

$$\left\{\begin{array}{rl} \mathbb{L}(u_\varepsilon,\bar g)\,[\bar v_{1,+}]=0 & D_{\eta_1,R_1} \\ \bar v_{1,+}=\psi_{1,+} & \partial D_{\eta_1,R_1} \end{array}\right. \tag{6.9}$$

(analogous problems for $\bar v_{1,-},\bar v_{2,+},\bar v_{2,-}$) and

$$\left\{\begin{array}{rcll} \mathbb{L}(u_\varepsilon,\bar g)\,[\,v_C] &=& 0 & \text{in} \quad C_\varepsilon\,, \\ v_C &=& \psi_{1,+} & \text{on} \quad \partial D_{\eta_1,R_1} \\ v_C &=& \psi_{1,-} & \text{on} \quad \partial D_{\eta_1,-R_1} \\ v_C &=& \psi_{2,+} & \text{on} \quad \partial D_{\eta_2,R_2} \\ v_C &=& \psi_{2,-} & \text{on} \quad \partial D_{\eta_2,-R_2}. \end{array}\right. \tag{6.10}$$

The Dirichlet data for the problems (6.9) and (6.10) must be chosen in a proper way. Namely, the choice of $\bar\psi\,:=(\,\psi_{1,+},\psi_{1,-},\psi_{2,+},\psi_{2,-})$ is dictated by the requirement that the function z has the correct regularity for being a solution on the whole M_ε. In fact z is certainly continuous for any choice of $\bar\psi$, but there may be a lack C^1-regularity across the interface $\partial D_{\eta_1,R_1}$. To avoid this situation, we impose the following C^1-matching condition

$$\begin{array}{l} \partial_{r_1}(v_{1,\pm}+\bar v_{1,\pm})_{|r_1=\pm R_1}=\partial_{r_1}v_{C_{|r_1=\pm R_1}} \\ \partial_{r_2}(v_{2,\pm}+\bar v_{2,\pm})_{|r_2=\pm R_2}=\partial_{r_2}v_{C_{|r_2=\pm R_2}}. \end{array} \tag{6.11}$$

To show that the *ansatz* above actually yields a good definition for z, we adopt the following strategy. First of all, we show that problems (6.9) and (6.10) have a unique solution with ε-uniform a priori estimate for generic Dirichlet data. Secondly, studying the behavior of the so called *Dirichlet to Neumann map*, we will calibrate the choice of the boundary data in such a way that conditions (6.11) are satified.

As we did for (6.7), we note that, modulo the use of the conformal equivariance property, (6.9) is of the same type as (5.37). Thus, Proposition 5.8 applies giving, for any Dirichlet datum $\psi_{1,+} \in C^{2,\beta}(\partial D_{\eta_1,R_1})$, a unique $\bar{v}_{1,+} \in C^{2,\beta}_{-\delta}(D_{\eta_1,R_1}) \oplus \mathcal{W}^+(D_{\eta_1,R_1})$ such that

$$\| \bar{v}_{1,+} \|_{C^{2,\beta}_{-\delta}(D_{\eta_1,R_1}) \oplus \mathcal{W}^+(D_{\eta_1,R_1})} \quad \leq \quad C \ \| \psi_{1,+} \|_{C^{2,\beta}(\partial D_{\eta_1,R_1})}. \tag{6.12}$$

Of course, the same holds for $\bar{v}_{1,-}, \bar{v}_{2,+}$, and $\bar{v}_{2,-}$.

To solve problem (6.10) we take advantage of the fact that for any fixed ε, the domain C_ε is compact and thus we can apply the Fredholm alternative. Hence, we have existence and uniqueness for problem (6.10), provided

$$\begin{cases} \mathbb{L}(u_\varepsilon, \bar{g})\,[v] = 0 & C_\varepsilon \\ v = 0 & \partial C_\varepsilon \end{cases}$$

only admits the trivial solution. To prove this, we are going to show that there exists a positive constant $B > 0$ independent of ε such that the *a priori* bound

$$\|v\|_{C^{2,\beta}_{\gamma - \frac{n-2k}{2k}}(C_\varepsilon)} \ \leq \ B \ \|f\|_{C^{0,\beta}_{\gamma-(n-2k)}(C_\varepsilon)} \tag{6.13}$$

is in force for solutions to

$$\begin{cases} \mathbb{L}(u_\varepsilon, \bar{g})\,[v] = f & C_\varepsilon \\ v = 0 & \partial C_\varepsilon \end{cases}$$

To prove (6.13), we use a blow-up argument similar to the one used in the proof of Proposition 6.2. The only difference is in the treatment of the case **1**. In particular, following the argument used in the above mentioned proof and one ends up with a function v_∞ such that $u_1^{-1}\, v_\infty$ is a non trivial smooth solution of the following boundary value problem

$$\begin{cases} \mathbb{L}(1\,, g_1)\,[u_1^{-1} v_\infty] \ = \ 0 & \text{in} \quad (-R_1,\, R_1) \times \mathbb{S}^{n-1} =: C_{0,1}\ , \\ v \ = \ 0 & \text{on} \quad \partial C_{0,1} \end{cases}$$

Thus, thanks to the conformal equivariance property (2.4) and Proposition 5.9, we infer that $v_\infty \equiv 0$, which is a contradiction. Thus, there exists a unique v_C solving (6.10) and such that

$$\|v_C\|_{C^{2,\beta}_{\gamma - \frac{n-2k}{2k}}(C_\varepsilon)} \ \leq \ B \ \|\bar{\psi}\|_{C^{\,2,\beta}(\partial C_\varepsilon)}\,. \tag{6.14}$$

Having at hand the functions $\bar{v}_{1,+}, \bar{v}_{1,-}, \bar{v}_{2,+}, \bar{v}_{2,-}$ and v_C, we can introduce the Dirichlet to Neumann maps for problems (6.9) and (6.10). For the exterior problem (6.9), we define the mapping $T:\, C^{2,\beta}(\partial C_\varepsilon) \longrightarrow C^{1,\beta}(\partial C_\varepsilon)$, whose action is given by

$$T:\, \bar{\psi} := (\psi_{1,+}, \psi_{1,-}, \psi_{2,+}, \psi_{2,-}) \longmapsto (-\partial_{r_1}\bar{v}_{1,+}, \partial_{r_1}\bar{v}_{1,-}; -\partial_{r_2}\bar{v}_{2,+}, \partial_{r_2}\bar{v}_{2,-})_{|_{r_1=\pm R_1; r_2=\pm R_2}}\,.$$

We notice that the action of T is diagonal in the sense that the Dirichlet datum defined on a connected component of the boundary ∂C_ε is mapped to a Neumann datum defined on the same connected component.

On the other hand, for the interior problem (6.10) we define the mapping $S_\varepsilon:\, C^{2,\beta}(\partial C_\varepsilon) \longrightarrow C^{1,\beta}(\partial C_\varepsilon)$ which acts in the following way

$$S_\varepsilon:\, \bar{\psi} \longmapsto (-\partial_{r_1} v_C, \partial_{r_1} v_C; -\partial_{r_2} v_C, \partial_{r_2} v_C)_{|_{r_1=\pm R_1; r_2=\pm R_2}}\,.$$

In terms of the operators T and S_ε the C^1-matching condition can be rewritten as

$$\begin{aligned} \big[(T - S_\varepsilon)[\bar{\psi}]\big]_{1,\pm} \quad &= \quad \pm\partial_{r_1} v_{1,\pm} \quad \text{on } \partial D_{\eta_1,\pm R_1} \\ \big[(T - S_\varepsilon)[\bar{\psi}]\big]_{2,\pm} \quad &= \quad \pm\partial_{r_2} v_{2,\pm} \quad \text{on } \partial D_{\eta_2,\pm R_2}. \end{aligned}$$

Thus, the *ansatz* for z actually yields a well defined correction if we prove that the above pseudodifferential problems is solvable. To this end, we are going to show that, up to choose the parameter ε small enough, the map

$$T - S_\varepsilon : C^{2,\beta}(\partial C_\varepsilon) \longrightarrow C^{1,\beta}(\partial C_\varepsilon)$$

is invertible. First of all, we notice that T and S_ε are well defined and, thanks to the *a priori* estimates on the solutions to problems (6.9) and (6.10), they are also ε-uniformly bounded. We prove now the following

Lemma 6.6. *As $\varepsilon \to 0$ the operator S_ε converges in norm to the operator S_0 defined as*

$$\begin{array}{rcl} S_0 \,:\, C^{2,\beta}(\partial C_{0,1}) \times C^{2,\beta}(\partial C_{0,2}) & \longrightarrow & C^{1,\beta}(\partial C_{0,1}) \times C^{1,\beta}(\partial C_{0,2}) \\ ((\psi_{1,+}, \psi_{1,-}), (\psi_{2,+}, \psi_{2,-})) & \longmapsto & \big((-\partial_{r_1} v_{C,1}, \partial_{r_1} v_{C,1})_{|_{r_1 = \pm R_1}}, (-\partial_{r_2} v_{C,2}, \partial_{r_2} v_{C,2})_{|_{r_2 = \pm R_2}} \big), \end{array}$$

where the function $v_{C,1}$ is the unique solution of

$$\left\{ \begin{array}{rcll} \mathbb{L}(1, g_1)\,[u_1^{-1} v_{C,1}] & = & 0 & \text{in} \quad C_{0,1}\ , \\ v_{C,1} & = & \psi_{1,+} & \text{on} \ \{R_1\} \times \mathbb{S}^{n-1} \\ v_{C,1} & = & \psi_{1,-} & \text{on} \ \{-R_1\} \times \mathbb{S}^{n-1} \end{array} \right. \tag{6.15}$$

Of course, an analogous problem characterise $v_{C,2}$.

Proof. The proof is by contradiction. Taking advantage of the uniform *a priori* bound (6.13), we can suppose that if the statement is not true, then there exist a sequence $(\varepsilon_j, \bar\psi_j)$ such that $\varepsilon_j \to 0$ and, for every $j \in \mathbb{N}$, $\|\bar\psi_j\|_{C^{2,\beta}(\partial C_{\varepsilon_j})} = 1$ and

$$\|(S_{\varepsilon_j} - S_0)\,(\bar\psi_j)\|_{C^{1,\beta}(\partial C_\varepsilon)} \ \geq\ 1/2\ . \tag{6.16}$$

Let v_C^j be the unique solution to problem (6.10) with $\bar\psi_j$ as Dirichlet boundary datum. Up to choose a subsequence, we may suppose that the boundary data $\bar\psi_j$ converge to some $\bar\psi_\infty$ in $C^2(\partial C_{0,1} \cup \partial C_{0,2})$. From the uniform *a priori* estimates (6.13) combined with the fact that $g_{\varepsilon_j} \to g_i$ in C^2 on the compact subsets of $D_{\eta_i} \setminus \{p_i\}$, we deduce that, up to a subsequence, also the functions v_C^j converge to some functions $v_{\infty,i}$ on the compact subsets of $C_{0,i} \setminus \{p_i\}$ with respect to the C^2-topology, $i = 1, 2$. Making use of the conformal equivariance property combined with the removable singularities Lemma 6.1 (which is in force since $\gamma < (n-2k)/k$ and $2 < 2k \leq n$), it is not difficult to see that $v_{\infty,i}$ can be extended through p_i to a smooth solution of problem (6.15) on the whole $C_{0,i}$, $i = 1, 2$. Since it is evident that problem (6.15) has a unique solution, the functions $v_{\infty,1}$ and $v_{\infty,2}$ must coincide with $v_{C,1}$ and $v_{C,2}$ respectively. As a consequence, their normal derivative at the boundary must agree. This contradicts (6.16). $\square$

In force of the lemma above, the invertibility of $T - S_\varepsilon$ is now a consequence of the invertibility of the limit pseudodifferential operator $T - S_0$. Now, since the spectrum of the limit operator $T - S_0$ is discrete it is sufficient to prove the injectivity of $(T - S_0)$. We reason by contradiction and we assume to have $\bar\psi = (\psi_{1,+}, \psi_{2,-}, \psi_{2,+}, \psi_{2,-}) \neq 0$ for which $(T - S_0)[\bar\psi] = 0$. The existence of such a boundary datum $\bar\psi$ implies the existence of a smooth function $\tilde v$ solving

$$\mathbb{L}(1, g_1)\,[u_1^{-1} \tilde v] \ =\ 0 \qquad \text{on} \quad D_{\eta_1}\ ,$$

and such that

$$\left\{ \begin{array}{rcl} \tilde v(R_1, \theta) & = & \psi_{1,+}(\theta) \\ \tilde v(-R_1, \theta) & = & \psi_{1,+}(\theta)\,. \end{array} \right.$$

Moreover, $\tilde v$ has the following form

$$\tilde v \ =\ \left\{ \begin{array}{lll} \bar v_{1,+} & \text{in} & D_{\eta_1, R_1} \\ v_{C,1} & \text{in} & C_{0,1} \\ \bar v_{1,-} & \text{in} & D_{\eta_1, -R_1}\ , \end{array} \right.$$

where we recall that $\bar{v}_{1,+}$ and $\bar{v}_{1,-}$ are solutions to problems of the type (6.9) and belong to $C^{2,\beta}_{-\delta}(D_{\eta_1,R_1}) \oplus \mathcal{W}^+(D_{\eta_1,R_1})$ and to $C^{2,\beta}_{-\delta}(D_{\eta_1,-R_1}) \oplus \mathcal{W}^-(D_{\eta_1,-R_1})$, respectively. Of course $\tilde{v}$ has the corresponding features on D_{η_2}, but since the situation is somehow symmetric we just focus on D_{η_1}. To reach the desired contradiction, we aim to show that $\tilde{v} \equiv 0$.

We perform the usual separation of variable, projecting the equation along the eigenfunction ϕ_j's of the Laplace-Beltrami operator on $(S^{n-1}, g_{S^{n-1}})$, having in mind that the high frequencies ($j \geq n+1$) and the low frequencies ($j = 0, \ldots, n$) will be treated separately

$$\tilde{v}\,(r_1, \theta) \;\; = \;\; \sum_{j=0}^{n} \tilde{v}^j(r_1)\,\phi_j(\theta) \, + \sum_{j=n+1}^{\infty} \tilde{v}^j(r_1)\,\phi_j(\theta) \; .$$

The high frequencies are easier to treat. In fact the deficiency space components are not present in this regime. Thus the $\tilde{v}^j$ are exponentially decreasing for $|r_1| \to +\infty$. Hence, the maximum principle forces them to be identically zero.

To obtain the same result for the low frequencies, we note that the functions $\tilde{v}^j$ for $j = 1, \ldots, n$ can be written as a linear combination of $\Phi^{j,+}_{\eta_1}(r_1, \theta)$ and $\Phi^{j,-}_{\eta_1}(r_1, \theta)$. Namely, for any $j = 1, \ldots, n$ there exist real numbers A_j, B_j such that

$$\tilde{v}^j(r_1) \, := \, A_j \left[\tfrac{n-2k}{2k} v_{D,\eta_1}(r_1) \, + \, \dot{v}_{D,\eta_1}(r_1) \right] e^{r_1} + B_j \left[\tfrac{n-2k}{2k} v_{D,\eta_1}(r_1) \, - \, \dot{v}_{D,\eta_1}(r_1) \right] e^{-r_1}.$$

Now, since $\tilde{v}^j \equiv \bar{v}^j_{1,+}$ for $r_1 > R_1$ and $\bar{v}_{1,+}$ has a component decaying like $e^{-\delta r_1}$ with $\delta > 1$ and the other one decaying like e^{-r_1} as $r_1 \to +\infty$, there holds that necessarily $A_j \, = \, 0$. The same argument used when $r_1 < -R_1$ shows that also $B_j \, = \, 0$. This implies that $\psi^j_{1,\pm} \, = \, 0$ for $j = 1, \ldots, n$.

The last case is when $j = 0$. As before, the function $\tilde{v}^0$ is a linear combination of the two Jacobi fields $\Phi^{0,-}_{\eta_1}$ and $\Phi^{0,+}_{\eta_1}$, namely, there exist real numbers A_0 and B_0 such that

$$v^0_\infty(r_1) \;\; = \;\; A_0\,\Phi^{0,+}_{\eta_1}(r_1) + \, B_0 \Phi^{0,-}_{\eta_1}(r_1) \; .$$

Comparing the asymptotic behavior at $\pm\infty$ of the expression above with the one prescribed by the constraints $\tilde{v}^0 = \bar{v}^0_{1,+}$ for $r_1 > R_1$ and $\tilde{v}^0 = v^0_{1,-}$ for $r_1 < R_1$, we get $A_0 = 0 = B_0$. As a consequence $\tilde{v}^0 \equiv 0$ and $\psi^0_\pm = 0$.

The conclusion is that $T - S_0$ is injective, hence invertible and for ε sufficiently small also $T - S_\varepsilon$ is invertible. As already observed, this implies that the correction z is well defined and thus the operator $\mathbb{L}(u_\varepsilon, \bar{g})$ is surjective on $C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \, \oplus \, \mathcal{W}(M_\varepsilon)$.

To conclude the proof of our statement, we need to establish the *a priori* estimate (6.6), which obviously implies injectivity. First of all we notice that thanks to (5.37), the solution w verifies, for some positive constant $C > 0$ independent of ε, the bound

$$\|w\|_{C^{2,\beta}_{-\delta}(D_{\eta_1,R_1}) \oplus \mathcal{W}^+(D_{\eta_1,R_1})} \;\leq\; C \; \left[\, \|f\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)} + \|w_{|\partial C_\varepsilon}\|_{C^{2,\beta}(\partial C_\varepsilon)} \, \right]$$

with analogous estimates on the other connected components of $M_\varepsilon \setminus C_\varepsilon$. Moreover, the same argument used to prove (6.14) implies that for ε sufficiently small

$$\|w\|_{C^{2,\beta}_{\gamma-\frac{n-2k}{2k}}(C_\varepsilon)} \;\leq\; C \; \left[\; \|f\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)} \; + \; \|w_{|\partial C_\varepsilon}\|_{C^{\,2,\beta}(\partial C_\varepsilon)} \; \right],$$

for some $C > 0$ independent of ε. Finally, it follows from standard interior Schauder estimates that the norm of w on ∂C_ε is uniformly bounded by the norm of f in a small neighborhood of ∂C_ε. Combining this remark with the last two estimates, it is now easy to obtain (6.6). This completes the proof. $\square$

7 Nonlinear analysis

In this last section we are going to correct the approximate solutions u_ε to exact solutions, provided the parameter ε is small enough. Again, for sake of simplicity, we limit ourself to the case $M_\varepsilon = D_{\eta_1} \sharp_\varepsilon D_{\eta_2}$, without discussing the minor changes needed for the general case. According to the linear analysis, it is natural to look for a correction lying in $C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \oplus \mathcal{W}(M_\varepsilon)$. Recalling that $\mathcal{W}(M_\varepsilon)$ is finite dimensional and identifying a function in this space with its coordinates with respect to the Jacobi fields basis, it is immediate to obtain the following isomorphism of Banach spaces

$$\begin{array}{rcl} \mathscr{I} \ : \ C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \oplus \mathcal{W}(M_\varepsilon) & \longrightarrow & C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \\ w \ = \ \hat{w} + \tilde{a}^{i,+}_j \Psi^{j,+}_{\eta_i} + \tilde{a}^{i,-}_j \Psi^{j,-}_{\eta_i} & \longmapsto & (\hat{w}\,, \boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}) \end{array}$$

where, for $i = 1, 2$ and $j = 0, \ldots, n$,

$$\tilde{a}^{i,+}_j \ := \ \chi_{R_i'}\, a^{i,+}_j \quad \text{and} \quad \tilde{a}^{i,-}_j \ := \ \chi_{-R_i'}\, a^{i,-}_j \quad \text{and} \quad \boldsymbol{a}^{i,\pm} \ := \ \big(a^{i,\pm}_0, \ldots, a^{i,\pm}_n\big)\,.$$

To describe our perturbation, we denote by $u_\varepsilon(\boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}, \cdot)$ the variation of u_ε with parameters $\boldsymbol{a}^{i,\pm}$, $i = 1, 2$, supported on $M_\varepsilon \setminus C_\varepsilon$, which is defined on D_{η_1, R_1} as

$$|\theta - (\tilde{a}^{1,+}_1, \ldots, \tilde{a}^{1,+}_n) e^{-r_1}|^{-\frac{n-2k}{2k}} v_{D,\eta_1}\big(\, r_1 + \log|\theta - (\tilde{a}^{1,+}_1, \ldots, \tilde{a}^{1,+}_n) e^{-r_1}| + \log(1 + \tilde{a}^{1,+}_0)\,\big)\ .$$

The definition of $u_\varepsilon(\boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}, \cdot)$ on the other connected component of $M_\varepsilon \setminus C_\varepsilon$ is analogous. We note *en passant* that the 'straight' approximate solution $u_\varepsilon(\cdot)$ corresponds to $\boldsymbol{a}^{i,\pm} = 0$, for $i = 1, 2$.

To get exact solutions for our nonlinear problem, we are led to look for $(\hat{w}, \boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}) \in C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that

$$\mathcal{N}(u_\varepsilon(\boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}, \cdot) + \hat{w}(\cdot)\,,\, \bar{g}\,) \ = \ 0\,. \tag{7.1}$$

A simple computation gives

$$\frac{d}{ds}\bigg|_{s=0} \mathcal{N}\left(u_\varepsilon(s\tilde{\boldsymbol{a}}^{1,+}, s\tilde{\boldsymbol{a}}^{1,-}, s\tilde{\boldsymbol{a}}^{2,+}, s\tilde{\boldsymbol{a}}^{2,-}, \cdot) + s\hat{w}(\cdot), \bar{g}\right) \ = \ \mathbb{L}(u_\varepsilon(\cdot), \bar{g})[w]\,,$$

where, according to (7.1),

$$w \ = \ \hat{w} + \tilde{a}^{i,+}_j \Psi^{j,+}_{\eta_i} + \tilde{a}^{i,-}_j \Psi^{j,-}_{\eta_i}\,. \tag{7.2}$$

This formula suggests that the perturbation of u_ε to an exact solution will involve a decaying term (through $\hat{w}$) together with small conformal variations of the former 'straight' approximate solution. Geometrically, these variations corresponds to translations along the Delaunay axis, changes of the Delaunay parameter η and 'bendings at infinity'.

Using a Taylor expansion, we can rewrite (7.1) as

$$\begin{array}{rcl} 0 & = & \mathcal{N}(u_\varepsilon(\tilde{\boldsymbol{a}}^{1,+}, \tilde{\boldsymbol{a}}^{1,-}, \tilde{\boldsymbol{a}}^{2,+}, \tilde{\boldsymbol{a}}^{2,-}, \cdot) + \hat{w}(\cdot), \bar{g}) \\ & = & \mathcal{N}(u_\varepsilon(\cdot), \bar{g}) \ + \ \mathbb{L}(u_\varepsilon(\cdot), \bar{g})[w] \ + \ \mathcal{Q}(u_\varepsilon(\cdot), \bar{g})\,[w, w]\ , \end{array}$$

where $\mathcal{Q}(u_\varepsilon(\cdot), \bar{g})\,[w, w]$ is the quadratic remainder. Thus, the fully nonlinear problem becomes equivalent to the following fixed point problem for $w = (\hat{w}, \boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}) \in C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$$w \ = \ \mathbb{L}(u_\varepsilon(\cdot), \bar{g})^{-1}\Big[\, -\mathcal{N}(u_\varepsilon(\cdot), \bar{g}) \ - \ \mathcal{Q}(u_\varepsilon(\cdot), \bar{g})[\, w, w\,]\,\Big]\,. \tag{7.3}$$

It is worth remarking that at first time one could have used a simple perturbation of the form $u_\varepsilon + w$ with w as (7.2). In fact, the first order expansion of $\mathcal{N}(u_\varepsilon + w, \bar{g})$ is also given by $\mathbb{L}(u_\varepsilon(\cdot), \bar{g})[w]$ and thus the linear analysis of the previous sections is still in force. On the other hand the components of the correction w along the Jacobi fields $\Psi^{0,\pm}_{\eta_i}$ are not necessarily infinitesimal with respect to u_ε when $|r_i| \to +\infty$, $i = 1, 2$, and this may possibly affect the completeness of the final solution. To avoid this problem we had to deal with perturbations of the form (7.1). Indeed the conformal equivariance of the σ_k-equation insures that the variations $u_\varepsilon(\boldsymbol{a}^{1,+}, \boldsymbol{a}^{1,-}, \boldsymbol{a}^{2,+}, \boldsymbol{a}^{2,-}, \cdot\,)$ are still complete exact solutions far away from the central body C_ε. Since the remaining part of the perturbation $\hat{w}$ is exponentially decaying at infinity, the completeness of the exact solutions is definitely guaranteed.

We denote by $\mathcal{P}$ the mapping $\mathcal{P} : C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon) \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ that associates to any w the right hand side of (7.3). In what follows, we will find the fixed point w as a limit of the sequence $\{w_i\}_{i\in\mathbb{N}}$ defined by means of the following Newton iteration scheme

$$
\begin{cases} w_0 \; := \; 0 \\ w_{i+1} \; := \; \mathcal{P}(w_i), \;\; i \in \mathbb{N}. \end{cases} \tag{7.4}
$$

To this end, we need some preparatory Lemmata. We state the following

Lemma 7.1. *There exists a positive constant $A = A(n,k) > 0$ such that for every $1 < \delta < \bar{\delta}\,(n,k)$ and $0 < \gamma < (n-2k)/k$ the proper error is estimated as*

$$
\| \, \mathcal{N}(u_\varepsilon, \bar{g}) \, \|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)} \;\; \leq \;\; A \, \varepsilon^{\gamma \frac{n-2k}{n}} . \tag{7.5}
$$

The proof of this result could be found in [4] to which we refer for all the details. Incidentally, we notice that the proof substantially uses that $\mathcal{N}(u_\varepsilon, \bar{g})$ is concentrated only on the neck region N_ε. Thus, even if we have to deal with noncompact manifolds, the computations needed to estimate this term are exactly of the same type of the computations in [4, Lemma 5.1].

In the following lemma we provide an estimate for the quadratc remainder outside the neck region.

Lemma 7.2. *There exists a positive constant $C = C(n,k,\delta,\gamma)$ such that for every $1 < \delta < \bar{\delta}\,(n,k)$ and $0 < \gamma < (n-2k)/k$ there holds*

$$
\|w\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} \leq \rho \;\; \Longrightarrow \;\; \|\mathcal{Q}(u_\varepsilon, \bar{g})[\,w, w\,]\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon \setminus N_\varepsilon)} \; \leq \; C\rho^2. \tag{7.6}
$$

Proof. Let us fix a positive ρ for which

$$
\|w\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} \leq \rho
$$

holds. We analyze the size of $\mathcal{Q}(u_\varepsilon, \bar{g})[\,w; w\,]$ according to the definition of the norm in $C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)$ in (6.2). In particular, we decompose $M_\varepsilon \setminus N_\varepsilon$ as $M_\varepsilon \setminus N_\varepsilon \; = \; D_{\eta_1,R_1} \cup D_{\eta_1,-R_1} \cup D_{\eta_2,R_2} \cup D_{\eta_2,-R_2} \cup C_\varepsilon \setminus N_\varepsilon$ and we prove that (7.6) holds on every component of the above decomposition. On this regard, let us notice that it will be sufficient to check (7.6) only on $C_\varepsilon \setminus (N_\varepsilon \cap D_{\eta_1})$ and on D_{η_1,R_1}.
We start with the analysis on D_{η_1,R_1}. We recall that in this region $\bar{g} \; = \; v_{D_{\eta_1}}^{4k/(n-2k)} \, g_{cyl}$. Hence, from the computational point of view it is more convenient to work with the cylindrical metric as a background metric. To this end we set $z \, := \, v_{D_{\eta_1}} \, w$, $\hat{z} \, := \, v_{D_{\eta_1}} \, \hat{w}$ and $z^\top \, := \, v_{D_{\eta_1}} w^\top$. Thus, by using the conformal equivariance property (2.4) and that $u_\varepsilon \, \equiv \, 1$ on D_{η_1,R_1} (see (4.1)), one has

$$
\mathcal{Q}(u_\varepsilon(\cdot), \bar{g}) \, [w; w] \; = \; v_{D_{\eta_1}}^{-\frac{2kn}{n-2k}} \, \mathcal{Q}(v_{D_{\eta_1}}(\cdot), g_{cyl}) \, [z; z], \quad w \; = \; v_{D_{\eta_1}}^{-1} \, z \, . \tag{7.7}
$$

Since $v_{D_{\eta_1}}$ is uniformly bounded from above and from below, one can deduce the desired estimate from its analogous for $\mathcal{Q}(v_{D_{\eta_1}}(\cdot), g_{cyl})\,[z\,;z]$. This last term can be expanded on D_{η_1,R_1} as

$$
\begin{aligned}
\mathcal{Q}(v_{D_{\eta_1}}(\cdot), g_{cyl})\,[z\,;z] \quad &= \quad \mathcal{Q}(v_{D_{\eta_1}}(\cdot), g_{cyl})\,[(\hat{z}, \tilde{\boldsymbol{a}}^{1,+}, 0,0,0)\,;(\hat{z}, \tilde{\boldsymbol{a}}^{1,+}, 0,0,0)] \\
&= \quad \int_0^1 \Big[\mathbb{L}(v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0,0,0,\cdot) + \tau\hat{z}(\cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0,0,0,\cdot), g_{cyl})\Big]\,[\hat{z}]\; d\tau \\
&\qquad + \Big[\mathbb{L}(v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0,0,0,\cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\cdot), g_{cyl})\Big]\,[\hat{z}] \\
&\qquad + \int_0^1 \Big[\mathbb{L}(v_{D_{\eta_1}}(\tau\tilde{\boldsymbol{a}}^{1,+}, 0,0,0,\cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\cdot), g_{cyl})\Big]\,[\tilde{a}_j^{1,+}\Psi_{\eta_1}^{j,1}]\; d\tau \\
&=: \quad Q_1 + Q_2 + Q_3\,.
\end{aligned}
\tag{7.8}
$$

To proceed, we recall that the linearization of $\mathcal{N}(\cdot, g_{cyl})$ around $v_{D_{\eta_1}}$ has the following general structure (see (2.1) and [17, Section 5])

$$
\mathbb{L}(v_{D_{\eta_1}}, g_{cyl}) \;=\; \mathbb{L}^0(v_{D_{\eta_1}}, g_{cyl}) + c_{n,k}v_{D_{\eta_1}}^{\frac{2kn}{n-2k}-1}, \tag{7.9}
$$

where $c_{n,k}$ is a computable positive constant and $\mathbb{L}^0(v_{D_{\eta_1}}, g_{cyl})$ is a second order differential operator with smooth coefficients of the following form

$$
\mathbb{L}^0(v_{D_{\eta_1}}, g_{cyl}) \;=\; \sum_{|\alpha|\leq 2} P_\alpha^{2k-1}(v_{D_{\eta_1}}, \nabla v_{D_{\eta_1}}, \nabla^2 v_{D_{\eta_1}})\,\partial^\alpha\,, \tag{7.10}
$$

where α is a multi-index and $P_\alpha^{2k-1}\,:\,\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{R}$ is an homogeneous polynomial of degree $2k-1$

$$
P_\alpha^{2k-1}(x,y,z) \;:=\; \sum_{\beta_0+|\beta_1|+|\beta_2|=2k-1} a_{\alpha,(\beta_0,\beta_1,\beta_2)}x^{\beta_0}y^{\beta_1}z^{\beta_2}.
$$

As a consequence, setting $\boldsymbol{h} := (h_0,h_1,h_2) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{\frac{n(n+1)}{2}}$ and expanding at first order P_α^{2k-1} one has

$$
P_\alpha^{2k-1}(x+h_0, y+h_1, z+h_2) - P_\alpha^{2k-1}(x,y,z) \;=\; DP_\alpha^{2k-1}(x,y,z)\cdot\boldsymbol{h} + O(|\boldsymbol{h}|^2). \tag{7.11}
$$

We have now all the ingredients to obtain the estimate on D_{η_1,R_1} for (7.8). First of all, we recall that (see (5.18) and (5.19))

$$
\begin{aligned}
\|\mathcal{Q}(v_{D_{\eta_1}}, g_{cyl})[\,z;z\,]\|_{C^{0,\beta}_{-\delta}(D_{\eta_1,R_1})} \quad &:= \quad \sup_{[R_1,+\infty)\times\mathbb{S}^{n-1}} (\cosh r_1)^\delta\,|\mathcal{Q}(v_{D_{\eta_1}}, g_{cyl})[\,z;z\,]| \\
&\quad + \quad \sup_{r_1\geq R_1+1} (\cosh r_1)^\delta\,[\,\mathcal{Q}(v_{D_{\eta_1}}, g_{cyl})[\,z;z\,]\,]_{C^{0,\beta}(\,(r_1-1,r_1+1)\times\mathbb{S}^{n-1}\,)}.
\end{aligned}
\tag{7.12}
$$

We will estimate separately the two terms in (7.12).

We start with the estimate of the weighted C^0 norm of Q_1. To this end, by applying (7.11) to the operator $\mathbb{L}^0$ and simply expanding at first order the remaining term in (7.9) we may decompose Q_1 into $Q_1 \,=\, q_{1,1} + q_{1,2}$ where, for any $(r_1,\theta) \,\in\, D_{\eta_1,R_1}$,

$$
q_{1,1} \;:=\; \int_0^1 \sum_{|\alpha|\leq 2} \big[DP_\alpha^{2k-1}(v_{D_{\eta_1}}, \nabla v_{D_{\eta_1}}, \nabla^2 v_{D_{\eta_1}})\cdot\tau\boldsymbol{h} + O(|\tau\boldsymbol{h}|^2)\big]\partial^\alpha\hat{z}\; d\tau \tag{7.13}
$$

and

$$
q_{1,2} \;:=\; \int_0^1 \big(d_{n,k}\, v_{D_{\eta_1}}^{\frac{2kn}{n-2k}-2}\,\tau\hat{z} + O(|\tau\hat{z}|^2)\big)[\hat{z}]\; d\tau, \quad d_{n,k} \;:=\; c_{n,k}\,\big(\tfrac{2kn}{n-2k}-1\big) \tag{7.14}
$$

where the vector $\boldsymbol{h}$ appearing in (7.13) has components $(\hat{z}, \nabla\hat{z}, \nabla^2\hat{z})$. Thus, it is immediate to obtain

$$\text{(7.15)} \qquad \|q_{1,1}\|_{C^0_{-\delta}(M_\varepsilon\setminus C_\varepsilon)} \,\leq\, C\,\|\hat{z}\|^2_{C^2_{-\delta}(M_\varepsilon\setminus C_\varepsilon)} \quad \text{and} \quad \|q_{1,2}\|_{C^0_{-\delta}(M_\varepsilon\setminus C_\varepsilon)} \,\leq\, C\,\|\hat{z}\|^2_{C^2_{-\delta}(M_\varepsilon\setminus C_\varepsilon)}$$

for some positive constant C, possibly depending on n, k, γ and δ.

Concerning Q_2, we preliminarly expand $v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0, 0, 0, \cdot\,)$ as (recall (5.15)-(5.17) and (5.35))

$$\text{(7.16)} \qquad v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0, 0, 0, \cdot\,) \,=\, v_{D_{\eta_1}}(\cdot) + \Psi^{j,+}_{\eta_1}\tilde{a}^{1,+}_j + O(\|z^\top\|^2_{\mathcal{W}(\mathcal{M}_\varepsilon)}).$$

Thus, splitting Q_2 into $Q_2 \,=\, q_{2,1} + q_{2,2}$, where

$$\text{(7.17)} \qquad q_{2,1} \,:=\, \big[\, \mathbb{L}^0\big(v_{D_{\eta_1}}(\cdot) + \Psi^{j,+}_{\eta_1}\tilde{a}^{1,+}_j + O(\|z^\top\|^2_{\mathcal{W}(\mathcal{M}_\varepsilon)})\,,\, g_{cyl}\,\big) \,-\, \mathbb{L}^0\big(\, v_{D_{\eta_1}}(\cdot), g_{cyl}\,\big)\,\big]\;[z]$$

and

$$\text{(7.18)} \qquad q_{2,2} \,:=\, c_{n,k}\,\big[\,(v_{D_{\eta_1}}(\cdot) + \Psi^{j,+}_{\eta_1}a^{1,+}_j + O(\|z^\top\|^2_{\mathcal{W}(\mathcal{M}_\varepsilon)}))^{\frac{2kn}{n-2k}-1} - c_{n,k}\, v_{D_{\eta_1}}^{\frac{2kn}{n-2k}-1}(\cdot)\,\big]\;[z]$$

Thus, by using (7.11) in (7.17) and expanding at first order in (7.18) we get

$$\text{(7.19)} \qquad \begin{aligned} \|q_{2,1}\|_{C^0_{-\delta}(M_\varepsilon\setminus C_\varepsilon)} \,&\leq\, C\,\|z^\top\|_{\mathcal{W}(M_\varepsilon)}\|z\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} \\ \|q_{2,2}\|_{C^0_{-\delta}(M_\varepsilon\setminus C_\varepsilon)} \,&\leq\, C\,\|z^\top\|_{\mathcal{W}(M_\varepsilon)}\|z\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)}\;, \end{aligned}$$

where the positive constant C possibly depends on n, k, γ and δ.

Now, we estimate Q_3. The estimate relies on the observation that Q_3 has compact support. To see this, let us show that

$$Q_3' \,:=\, \big[\,\mathbb{L}(v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0, 0, 0, \cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\cdot), g_{cyl})\,\big]\,[\tilde{a}^{1,+}_j\Psi^{j,+}_{\eta_1}]$$

has indeed compact support on D_{η_1,R_1}. We may decompose Q_3' as

$$\begin{aligned} Q_3' \quad &= \quad \big[\,\mathbb{L}(v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0, 0, 0, \cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot), g_{cyl})\big]\,[\tilde{a}^{1,+}_j\Psi^{j,+}_{\eta_1}] \\ &\quad\; + \big[\,\mathbb{L}(v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\cdot), g_{cyl})\,\big]\,[(\tilde{a}^{1,+}_j - a^{1,+}_j)\Psi^{j,+}_{\eta_1}] \\ &\quad\; + \big[\,\mathbb{L}(v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot), g_{cyl}) - \mathbb{L}(v_{D_{\eta_1}}(\cdot), g_{cyl})\,\big]\,[a^{1,+}_j\Psi^{j,+}_{\eta_1}] \\ &=: \quad q'_{3,1} + q'_{3,2} + q'_{3,3}\;. \end{aligned}$$

Now, $q'_{3,3} \,\equiv\, 0$ in D_{η_1,R_1}. In fact, we observe that

$$q'_{3,3} \,=\, \int_0^1 D^2\mathcal{N}(v_{D_{\eta_1}}(s\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot), g_{cyl})\,[a^{1,+}_j\Psi^{j,+}_{\eta_1}, a^{1,+}_i\Psi^{i,+}_{\eta_1}]\;ds\,,$$

and that for any $(r_1, \theta) \in D_{\eta_1,R_1}$

$$\mathcal{N}(v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, r_1, \theta), g_{cyl}) \,=\, 0\,.$$

Now, since $\tilde{a}^{1,+}_j - a^{1,+}_j \,=\, a^{1,+}_j(\chi_{R_1'} - 1)$ has compact support, it turns out that also $q'_{3,2}$ is compactly supported. To see that the remaining term $q'_{3,1}$ has compact support, we first expand $v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0, 0, 0, \cdot)$ around $v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot)$ as

$$v_{D_{\eta_1}}(\tilde{\boldsymbol{a}}^{1,+}, 0, 0, 0, \cdot) \,=\, v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot) + \frac{\partial v_{D_{\eta_1}}(\boldsymbol{a}^{1,+}, 0, 0, 0, \cdot)}{\partial a^{1,+}_j}(\tilde{a}^{1,+}_j - a^{1,+}_j) \,+\, O(|\tilde{\boldsymbol{a}}^{1,+} - \boldsymbol{a}^{1,+}|^2).$$

As before, note that $\frac{\partial v_{D_{\eta_1}}(\boldsymbol{a}^{1,+},0,0,0,\cdot)}{\partial a_j^{1,+}}(\tilde a_j^{1,+} - a_j^{1,+}) + O(|\tilde{\boldsymbol{a}}^{1,+} - \boldsymbol{a}^{1,+}|^2)$ has compact support. Now, recalling (7.10) and (7.11) and expanding at first order the potential term in (7.9), it is not difficult to get

$$|q'_{3,1}| \le C\,\|z^\top\|_{\mathcal{W}(M_\varepsilon)} \left| \frac{\partial v_{D_{\eta_1}}(\boldsymbol{a}^{1,+},0,0,0,\cdot)}{\partial a_j^{1,+}}(\tilde a_j^{1,+} - a_j^{1,+}) + O(|\tilde{\boldsymbol{a}}^{1,+} - \boldsymbol{a}^{1,+}|^2) \right|,$$

which clearly implies that $q'_{3,1}$ is compactly suported.

We can now give the desired estimate for Q_3. In particular, thanks to the above computations, it is evident that we can equivalently estimate the C^0 norm of Q_3 instead of its $C^0_{-\delta}$ norm. To obtain this estimate we reason as before. Using (7.16), (7.10) and (7.11) and expanding at first order the potential term in (7.9) it is standard to get

$$\sup_{(r_1,\theta)\in(R_1,+\infty)\times\mathbb{S}^{n-1}} |Q_3|(r_1,\theta) \le C\,\|z^\top\|^2_{\mathcal{W}(M_\varepsilon)}. \tag{7.20}$$

Thus, collecting (7.15), (7.19), (7.20) and recalling that $v_{D_{\eta_1}}$ is uniformly bounded from below and from above, we get (see (7.7)) the weighted C^0 estimate for $\mathcal{Q}(u_\varepsilon,\bar g)[w;w]$ on D_{η_1,R_1}, namely the following

$$\sup_{[R_1,+\infty)\times\mathbb{S}^{n-1}} (\cosh r_1)^\delta\,|\mathcal{Q}(u_\varepsilon,\bar g)[w;w]|\,(r_1,\theta) \le C\rho^2. \tag{7.21}$$

Now, we turn our attention to the estimate for the Hölder quotients. We will use two different strategies. In particular, for the terms Q_1 and Q_2, we will estimate directly their Hölder quotient. For Q_3 we will estimate its weighted C^1 norm. This is possible thanks to its particular structure. More precisely, it is possible to obtain a weighted C^1 estimate by relying, loosely speaking, on the regularity of the Jacobi fields and of the $\tilde{\boldsymbol{a}}^{1,+} := \chi_{R_1'}\boldsymbol{a}^{1,+}$.

We start with the estimate of the term Q_3. By first using (7.16) and then expanding at first order the coefficients of the linearized operator as in (7.11), it is sufficient to get a weighted C^0 estimate for

$$\nabla\,\Big[\sum_{|\alpha|\le 2}\big[DP_\alpha^{2k-1}(v_{D_{\eta_1}},\nabla v_{D_{\eta_1}},\nabla^2 v_{D_{\eta_1}})\cdot\boldsymbol{h} + O(|\boldsymbol{h}|^2)\big]\,\partial^\alpha(\tilde a_j^{1,+}\Psi_{\eta_1}^{j,+})\Big]$$

and for

$$\nabla\,\Big[\big(d_{n,k}\,v_{D_{\eta_1}}(\cdot)^{\frac{2kn}{n-2k}-2}\,h_0 + O(|h_0|^2)\big)\,[\tilde{\boldsymbol{a}}_j^{1,+}\Psi_{\eta_1}^{j,1}]\Big], \quad d_{n,k} := c_{n,k}\big(\tfrac{2kn}{n-2k}-1\big),$$

where the vector $\boldsymbol{h}$ has components $h_i = \nabla^i(\tilde a_j^{1,+}\Psi_{\eta_1}^{j,+} + O(\|w^\top\|^2_{\mathcal{W}(\mathcal{M}_\varepsilon)})$, for $i = 0,1,2$. We will outline only the estimate for the first term. A similar argument applies to the second. First of all, from the definition of $\tilde{\boldsymbol{a}}^{1,+} := \chi_{R_1'}\boldsymbol{a}^{1,+}$ (recall that the cut off function χ_{R_1} is smooth and bounded with its derivatives) and from the definition of the Jacobi fields, we easily get

$$|\nabla\,\partial^\alpha(\tilde a_j^{1,+}\Psi_{\eta_1}^{j,+})| \le C\,\|z^\top\|_{\mathcal{W}(\mathcal{M}_\varepsilon)}, \quad |\alpha|\le 2$$

Thus, since $\sup_{r_1\ge R_1}|\nabla^i v_{D_{\eta_1}}| \le C$, for $i=0,1,2,3$, we get

$$\sup_{(r_1,\theta)\in[R_1,+\infty)\times\mathbb{S}^{n-1}} |\nabla Q_3|(r_1,\theta) \le C\,\|z^\top\|^2_{\mathcal{W}(\mathcal{M}_\varepsilon)},$$

which implies, together with (7.20),

$$\|Q_3\|_{C^{0,\beta}_{-\delta}(D_{\eta_1})} \le C\|z^\top\|^2_{\mathcal{W}(\mathcal{M}_\varepsilon)}. \tag{7.22}$$

Now, we estimate the Hölder quotients for Q_1 and Q_2. We start with Q_1. As we did for the C^0 estimate, we split Q_1 into $Q_1 = q_{1,1} + q_{1,2}$. We will detail only the estimate for the Hölder quotient of $q_{1,1}$, the

one for $q_{1,2}$ being completely analogous. Recalling that $\boldsymbol{h}$ is the vector with components $h_i = \nabla^i_{g_{cyl}} \hat{z}$, for $i = 0,1,2$, we can write

$$
\begin{aligned}
q_{1,1}(r,\theta) - q_{1,1}(r',\theta') \;=\; & \int_0^1 \sum_{|\alpha|\leq 2} \Big[DP^{2k-1}_{\alpha}(v_{D_{\eta_1}}(r), \nabla v_{D_{\eta_1}}(r), \nabla^2 v_{D_{\eta_1}}(r)) \, [\tau \boldsymbol{h}(r,\theta)] \; \partial^\alpha \hat{z}(r,\theta) \\
& \qquad - DP^{2k-1}_{\alpha}(v_{D_{\eta_1}}(r'), \nabla v_{D_{\eta_1}}(r'), \nabla^2 v_{D_{\eta_1}}(r')) \, [\tau \boldsymbol{h}(r',\theta')] \; \partial^\alpha \hat{z}(r',\theta') \Big] \, d\tau \\
& + \int_0^1 \sum_{|\alpha|\leq 2} \Big[G(\tau \boldsymbol{h}(r,\theta)) \, \partial^\alpha \hat{z}(r,\theta) - G(\tau \boldsymbol{h}(r',\theta')) \, \partial^\alpha \hat{z}(r',\theta') \Big] \, d\tau, \qquad (7.23)
\end{aligned}
$$

where G is a smooth function such that $G(\boldsymbol{v}) = O(|\boldsymbol{v}|^2)$ and $DG(\boldsymbol{v}) = O(|\boldsymbol{v}|)$. Using the short notation

$$
A_\alpha(r,\theta) := DP^{2k-1}_{\alpha}(v_{D_{\eta_1}}(r), \nabla v_{D_{\eta_1}}(r), \nabla^2 v_{D_{\eta_1}}(r)) \,,
$$

we split integrand of the first summand in the expression above into

$$
\begin{aligned}
\sum_{|\alpha|\leq 2} \Big[& A_\alpha(r,\theta) \, [\tau \boldsymbol{h}(r,\theta)] \cdot [\, \partial^\alpha \hat{z}(r,\theta) - \partial^\alpha \hat{z}(r',\theta') \,] + A_\alpha(r',\theta') \, [\tau \boldsymbol{h}(r,\theta) - \tau \boldsymbol{h}(r',\theta')] \cdot \partial^\alpha \hat{z}(r',\theta') \\
& + \big[A_\alpha(r,\theta) - A_\alpha(r',\theta') \big] \, [\tau \boldsymbol{h}(r,\theta)] \cdot \partial^\alpha \hat{z}(r',\theta') \Big] \,.
\end{aligned}
$$

Using the fact that $(\cosh r)^{-\delta} < 1$, for $\delta > 0$ it is now easy to bound the weighted Hölder quotient of each term by a constant times $\|\hat{z}\|^2_{C^{2,\beta}_{-\delta}(D_{\eta_1,R_1})}$. Applying the same reasoning to the second summand in (7.23) and to the term $q_{1,2}$, one concludes that

$$
\sup_{r_1 \geq R_1+1} (\cosh r_1)^\delta \, [Q_1]_{C^{0,\beta}(\,(r_1-1, r_1+1)\times \mathbb{S}^{n-1}\,)} \;\leq\; C \, \|\hat{z}\|^2_{C^{2,\beta}_{-\delta}(D_{\eta_1,R_1})} \,. \tag{7.24}
$$

Using the same arguments, one can deduce the same type of estimate for the Hölder quotient of Q_2, namely

$$
\sup_{r_1 \geq R_1+1} (\cosh r_1)^\delta \, [\,Q_2,]_{C^{0,\beta}(\,(r-1, r+1)\times \mathbb{S}^{n-1}\,)} \;\leq\; C \|\hat{z}\|_{C^{2,\beta}_{-\delta}(M_\varepsilon \setminus C_\varepsilon)} \|z\|_{C^{2,\beta}_{-\delta}(M_\varepsilon \setminus C_\varepsilon) \oplus \mathcal{W}(M_\varepsilon)}. \tag{7.25}
$$

Thus, combining (7.21) with (7.22), (7.24) and (7.25) and recalling that $v_{D_{\eta_1}}$ is uniformly bounded from above and from below, we obtain

$$
\|\mathcal{Q}(u_\varepsilon, \bar{g})[\,w; w\,]\|_{C^{0,\beta}_{-\delta}(D_{\eta_1,R_1})} \;\leq\; C\rho^2. \tag{7.26}
$$

As anticipated, the estimates of $\mathcal{Q}(u_\varepsilon(\cdot), \bar{g})[\,\cdot\,;\,\cdot\,]$ on the other ends $D_{\eta_1,-R_1}, D_{\eta_2,R_2}, D_{\eta_2,-R_2}$ clearly follows from a similar argument. Thus, (7.26) actually becomes

$$
\|\mathcal{Q}(u_\varepsilon, \bar{g})[\,w; w\,]\|_{C^{0,\beta}_{-\delta}(M_\varepsilon \setminus C_\varepsilon)} \;\leq\; C\rho^2. \tag{7.27}
$$

Finally, it remains to estimate $\mathcal{Q}(u_\varepsilon(\cdot), \bar{g})$ on $C_\varepsilon \setminus N_\varepsilon$. Since on this region $u_\varepsilon(\tilde{\boldsymbol{a}}^{1,+}, \tilde{\boldsymbol{a}}^{1,-}, \tilde{\boldsymbol{a}}^{2,+}, \tilde{\boldsymbol{a}}^{2,-}, \cdot)$ coincides with $u_\varepsilon(\cdot)$, it turns out that the quadratic remainder can be written as

$$
\begin{aligned}
\mathcal{Q}(u_\varepsilon(\cdot), \bar{g}) \, [w; w] \;&=\; \mathcal{Q}(u_\varepsilon(\cdot), \bar{g}) \, [\hat{w}; \hat{w}] \\
&=\; \int_0^1 \big[\mathbb{L}(u_\varepsilon(\cdot) + \tau \hat{w}(\cdot), \bar{g}) - \mathbb{L}(u_\varepsilon(\cdot), \bar{g}) \,\big] [\hat{w}] \; d\tau \,.
\end{aligned}
$$

Thus, using an argument similar to the one used above (alternatively, one may refer to [4]), we have

$$
\|\mathcal{Q}(u_\varepsilon, \bar{g})[\,w; w\,]\|_{C^{0,\beta}(C_\varepsilon \setminus N_\varepsilon)} \;\leq\; C\rho^2.
$$

Thus, the lemma is proven. $\square$

We are now in the position to conclude the proof of Theorem 1. We need to prove that the sequence of the solutions to the iterative scheme (7.4) (which exist thanks to Proposition 6.6) is equibounded in $C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)$. We start with the estimate on w_1. Thanks to the uniform *a priori* estimate (6.6) for the linearized problem and to the estimate of the proper error term in Lemma 7.1, we immediately have

$$\|w_1\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)}\ \le\ AL\varepsilon^{(\gamma+2)\frac{n-2k}{n}}, \tag{7.28}$$

where the constant $L=L(\delta,\gamma,n,k)$ denotes the uniform bound on the norm of $\mathbb{L}(u_\varepsilon(\cdot),\bar g)^{-1}$, while the constant $A=A(\delta,\gamma,n,k)$ is the constant appearing in Lemma 7.1.

We proceed with the estimate of w_2. From the very definition of w_2, we have

$$\begin{aligned}\|w_2\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} &\le L\,\|\,\mathcal{N}(u_\varepsilon,\bar g)\,+\,\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)}\\ &\le AL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,+\,L\,\|\,\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)}.\end{aligned}\tag{7.29}$$

Thus, we need to estimate the quadratic remainder. Recalling the definition of the global weighted norm (6.2) in M_ε, we have the following

$$\begin{aligned}\|\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)} =&\ \|\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta}(M_\varepsilon\setminus N_\varepsilon)}+\sup_{N_\varepsilon}(\varepsilon\cosh t)^{\gamma-(n-2k)}\big|\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\big|\\ &+\sup_{t\in(\log\varepsilon,-\log\varepsilon)}\ (\varepsilon\cosh t)^{\gamma-(n-2k)}\,[\,\mathcal{Q}(u_\varepsilon,\bar g)[w_1]\,]_{C^{0,\beta}((t-1,t+1)\times\mathbb{S}^{n-1})}\,.\end{aligned}$$

Thanks to Lemma 7.2 and to estimate (7.28), the first term is estimated in this way

$$\|\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta}(M_\varepsilon\setminus N_\varepsilon)}\ \le\ C\,\|w_1\|^2_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)}\le ACL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}.$$

We consider now the second term (the term containing the Hölder quotients will be then estimated in the same way, see [4]). On the neck region N_ε, $\mathcal{Q}(u_\varepsilon(\cdot),\bar g)[w_1;w_1]$ has this form

$$\begin{aligned}\mathcal{Q}(u_\varepsilon(\cdot),\bar g)\,[w_1;w_1] &= \mathcal{Q}(u_\varepsilon(\cdot),\bar g)\,[\hat w_1;\hat w_1]\\ &= \int_0^1\big[\mathbb{L}(u_\varepsilon(\cdot)+\tau\hat w(\cdot),\bar g)-\mathbb{L}(u_\varepsilon(\cdot))\big][\hat w_1]\,d\tau.\end{aligned}$$

Thus, by exploiting the structure of $\mathbb{L}(u_\varepsilon(\cdot),\bar g)$ given in (7.9), using a first order expansion and recalling also (7.28) we have that there exists a positive constant independent of ε such that

$$\begin{aligned}(\varepsilon\cosh t)^{\gamma-(n-2k)}\,\big|\mathcal{Q}(u_\varepsilon(\cdot),\bar g)[w_1;w_1]\big| &\le C\,(\varepsilon\cosh t)^{\gamma-(n-2k)}(\varepsilon\cosh t)^{(2k-2)\frac{n-2k}{2k}}(\varepsilon\cosh t)^{-2\gamma+\frac{n-2k}{k}}\,\|\hat w_1\|^2_{C^{2,\beta}_{\gamma-\frac{n-2k}{2k}}(N_\varepsilon)}\\ &\le ACL\,\varepsilon^{-\gamma}\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,\|w_1\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)},\end{aligned}\tag{7.30}$$

where we have used also the fact that, for $j=0,1,2$, $\nabla^j_{\bar g}(u_\varepsilon)\,=\,O((\varepsilon\cosh t)^{\frac{n-2k}{2k}})$ on the neck region. Now, since $-\gamma+(\gamma+2)\frac{n-2k}{n}\,>\,0$, for any $\gamma\in(0,\frac{n-2k}{k})$, we have that, setting

$$B\,:=\,AL^2C\varepsilon^{-\gamma}\varepsilon^{(\gamma+2)\frac{n-2k}{n}},$$

there exists a positive number $\varepsilon_0\,=\,\varepsilon_0(\delta,\gamma,n,k)$ such that, for any $\varepsilon\in(0,\varepsilon_0]$, we can choose $B\,\le\,\frac14$. Consequently, the estimate (7.29) for w_2 becomes

$$\begin{aligned}\|w_2\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} &\le L\,\|\,\mathcal{N}(u_\varepsilon,\bar g)\,+\,\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)}\\ &\le AL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,+\,L\,\|\,\mathcal{Q}(u_\varepsilon,\bar g)[w_1;w_1]\|_{C^{0,\beta}_{-\delta,\gamma-(n-2k)}(M_\varepsilon)}\\ &\le AL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,+\,\tfrac14\,\|w_1\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)}.\end{aligned}$$

Then, we can iterate the above estimate obtaining, for $j \geq 1$,

$$\|w_{j+1}\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} \leq AL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,a_{j+1}, \tag{7.31}$$

where the sequence a_j is inductively defined as

$$\begin{cases} a_1 & := 1 \\ a_{j+1} & := 1 + \frac{1}{4}\,a_j^2, \quad j \in \mathbb{N}. \end{cases}$$

Now, a straightforward induction argument shows that $\sup_j a_j \leq 2$, thus estimate (7.31) becomes

$$\|w_j\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} \leq 2AL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,. \tag{7.32}$$

The previous estimate, combined with the fact that the embedding $C^{2,\beta}_{-\delta}(M_\varepsilon) \longrightarrow C^2_{-\delta'}(M_\varepsilon)$ is compact for any $\delta' < \delta$ (see [18, Chapter 12]), implies (up to a subsequence) the convergence in $C^2_{-\delta'}(M_\varepsilon)$ of w_i to a fixed point $w_\varepsilon = (\hat{w}_\varepsilon, \boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-})$ of the problem (7.3).

Thanks to the canonical identification (7.1) we will write, with a little abuse of notation, $w_\varepsilon = \hat{w}_\varepsilon + \tilde{a}_j^{\varepsilon,i,+}\Psi^{j,+}_{\eta_i} + \tilde{a}_j^{\varepsilon,i,-}\Psi^{j,-}_{\eta_i}$. Since (7.32) is uniform with respect to j, w_ε verifies

$$\|w_\varepsilon\|_{C^{2,\beta}_{-\delta,\gamma-\frac{n-2k}{2k}}(M_\varepsilon)\oplus\mathcal{W}(M_\varepsilon)} \leq 2AL\,\varepsilon^{(\gamma+2)\frac{n-2k}{n}}\,. \tag{7.33}$$

We claim now that there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in [0, \varepsilon_0)$ the exact solutions

$$u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot) + \hat{w}_\varepsilon(\cdot)$$

are positive. To see this fact we first observe that up to choose ε sufficiently small, the function $y_\varepsilon := u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot)$ is positive everywhere by definition. Secondly, since $\hat{w}_\varepsilon$ decays faster than y_ε along the complete ends, the exact solution $y_\varepsilon + \hat{w}_\varepsilon$ is certainly positive outside of a compact region $K_0 \subset M_\varepsilon$. Hence, since (7.33) implies that

$$\|y_\varepsilon^{-1}\hat{w}_\varepsilon\|_{C^{2,\beta}(K_0)} \leq C\varepsilon^{-\gamma+(\gamma+2)\frac{n-2k}{n}}, \tag{7.34}$$

there holds that $u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot) + \hat{w}_\varepsilon = y_\varepsilon + \hat{w}_\varepsilon = y_\varepsilon(1 + y_\varepsilon^{-1}\hat{w}_\varepsilon) > 0$.

For $\varepsilon \in (0, \varepsilon_0]$ we set

$$\widetilde{g_\varepsilon} := (u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot) + \hat{w}_\varepsilon)^{\frac{4k}{n-2k}}\,\bar{g}.$$

The above considerations imply that $\widetilde{g_\varepsilon}$ is the metric sought. We recall that the completeness of these metrics is a consequence of the decaying of $\hat{w}_\varepsilon$ on the ends of M_ε and of the fact that $u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot)$ is, by construction, a complete solution to the σ_k-equation, locally on the end.

Moreover, the family of metrics $\widetilde{g_\varepsilon}$ converges to the initial metric g_i with respect to the C^2 topology on every compact subset of $D_{\eta_i} \setminus \{p_i\}$, for $i = 1, 2$. This is evident on the four ends $D_{\eta_i,\pm R_i}$. In fact on these regions we have $\bar{g} = g_i$ and (7.33) implies that on every compact subset of $D_{\eta_i,\pm R_i}$ $u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot) + \hat{w}_\varepsilon \longrightarrow u_\varepsilon(\cdot) = 1$ in C^2 as $\varepsilon \to 0$. To see that $\widetilde{g_\varepsilon} \longrightarrow g_i$ on $D_{\eta_i} \setminus \{p_i\} \cap C_\varepsilon$, we recall, as before, that $u_\varepsilon(\boldsymbol{a}^{\varepsilon,1,+}, \boldsymbol{a}^{\varepsilon,1,-}, \boldsymbol{a}^{\varepsilon,2,+}, \boldsymbol{a}^{\varepsilon,2,-}, \cdot) \equiv u_\varepsilon$ on C_ε. Thus, the metric $\widetilde{g_\varepsilon}$ could be written as

$$\widetilde{g_\varepsilon} = (1 + u_\varepsilon^{-1}w_\varepsilon)^{\frac{4k}{n-2k}}\,g_\varepsilon.$$

Now, since by construction on every compact subset of $D_{\eta_i} \setminus \{p_i\} \cap C_\varepsilon$ the metric g_ε converges to the initial metric g_i in C^2 as $\varepsilon \to 0$, (7.34) implies that also the exact solutions $\widetilde{g_\varepsilon}$ tend to the initial metric g_i with respect to the C^2–topology on the compact subsets of $D_{\eta_i} \setminus \{p_i\} \cap C_\varepsilon$, for $i = 1, 2$, as $\varepsilon \to 0$. This concludes the proof of Theorem 1.

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